CONCERNING ABSOLUTELY CONTINUOUS FUNCTIONS.

BY PROFESSOR M. B. PORTER.

In a paper "Sulle funzioni integrali" published in 1905 in the *Atti della R. Accademia delle Scienze di Torino,* Vitali defined an important class of functions of limited variation to which he gave the name of *absolutely* continuous functions. He defines these functions as follows:

Let $F(x)$ be a finite function of the real variable x in an interval (a, b), where $a < b$, and let (α, β) be a partial interval of (a, b), $a \leqq \alpha < \beta \leqq b$. Call $F(\beta) - F(\alpha)$ the *increment* of $F(x)$ in (α, β). Call the sum of such increments, if it is finite and determinate, over a group of distinct (α, β)-intervals, the *increment of $F(x)$ in this group;* then, if for every $\sigma > 0$ there exists a $\mu > 0$ such that the modulus of the increment of $F(x)$ over every group of intervals of sum less than μ is less than σ, then $F(x)$ is said to be absolutely continuous in (a, b). Vitali then shows that $F(x)$ is a continuous function of limited variation, while continuous functions of limited variation are *not* all absolutely continuous, and establishes among others the following important

THEOREM. $F(x) - F(a) \equiv \int_a^x \Lambda F(x)dx$, *where* $\int \Lambda F$ *denotes the Lebesgue integral of one of the derivates of $F(x)$; and absolutely continuous functions are the only ones possessing this property.**

Lebesgue had already shown that the derivates of continuous functions of limited variation are summable and that in certain special cases the Lebesgue integral is the primitive function. Vitali's necessary and sufficient condition completes Lebesgue's theory in an important particular and shows that absolutely continuous functions constitute an important generalization of the class of analytic functions, and just as analytic functions can frequently be defined by general descriptive properties it is to be expected that such properties might exist for Vitali's functions. It is the purpose of this paper to show that

* For a proof of this theorem see Vallée Poussin's Cours d'Analyse, Tome 1, § 265, 3d edition.

THEOREM I: *A continuous function of limited variation, whose derivates are infinite only over a denumerable point set E, is absolutely continuous.*

We first remark that in virtue of a theorem of W. H. Young* the set E will always be either denumerable or of the power of the continuum. The inverse theorem is not true, for we shall show, by means of examples, that absolutely continuous functions exist with infinite derivates over any assigned point set of measure zero.

To prove Theorem I, first consider the case where $F(x)$ is monotone increasing. Then†

$$\ddagger \Delta F(x) \geqq \Delta \int_a^x \Delta F(x)dx,$$

so that $\Delta \int_a^x \Delta F(x)dx$ is finite whenever $\Delta F(x)$ is finite. We have now but to apply Vallée Poussin's generalization of Scheeffer's theorem (page 101, ibid.) to see that

$$\int_a^x \Delta F(x)dx \equiv F(x) - F(a),$$

which proves that $F(x)$ is absolutely continuous.

To prove the general case we have but to note that

$$\int_a^x |\Delta F(x)|\, dx \geqq \int_a^x \Delta F(x)dx \geqq -\int_a^x |\Delta F(x)|\, dx$$

and again apply Scheeffer's theorem.

As a corollary, we have that if the derivates become infinite over a reducible set, $F(x)$ is absolutely continuous.

We shall now show that no further generalization is possible. To do this consider the function§ $\phi(x)$ defined as follows:

Starting with any null set E and a number c, we take a set of intervals α_{ij} such that

$$\beta_i = \sum_{j=1}^{j=\infty} \alpha_{ij} \leqq \frac{c}{2^i},$$

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* *Arkiv för Matematik, Astronomi och Fysik*, vol. 1, Stockholm, 1903, or see Hobson's Theory of Functions of a Real Variable, p. 285, for an account of Young's work.

† Cf. Vallée Poussin: Cours d'Analyse, vol. 1, p. 275.

‡ ΔF denotes the upper right-hand derivate of $F(x)$.

§ See Vallée Poussin, loc. cit., vol. 1, p. 100, bottom.

so that

$$\sum_1^\infty \beta_i \leqq c.$$

The points of E are now enclosed in the open intervals α_{ij}, so that each point is inside of an infinite number of intervals, and $\phi(x)$ is defined to be the sum of all the α-intervals or parts thereof which lie to the left of x.

Thus $\phi(x)$ is monotone and can easily be shown to be absolutely continuous as follows:

If $i + j = N$ is chosen sufficiently large, the $\phi_N(x)$ formed for this finite set of intervals will be absolutely continuous and as near as we please to $\phi(x)$ for all values of x. Hence $\phi(x)$ is absolutely continuous.

If the set E is not an *inner limiting* set, the set $E'' = E + \overline{E}$, which lies inside an infinite number of α intervals, will be such a set, and $\phi(x)$ will have an infinite derivative at all the points of E'' and no others. The set E may itself be an inner limiting set, in which case $\overline{E} = 0$.

It would be interesting to determine whether all absolutely continuous functions are of the form

$$F(x) + \phi(x),$$

where $F(x)$ has limited derivates.

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ON THE REPRESENTATION OF NUMBERS IN THE FORM $x^3 + y^3 + z^3 - 3xyz$.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, August 3, 1915.)

IF by $g(x, y, z)$ we denote the form

$$(1)\quad g(x, y, z) = x^3 + y^3 + z^3 - 3xyz$$
$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx),$$

then it is well known that

$$g(x, y, z) \cdot g(u, v, w) = g(xu + yw + zv,\ xv + yu + zw,$$
$$xw + yv + zu).$$

By interchanging the rôles of v and w, we also have

$$g(x, y, z) \cdot g(u, v, w) = g(xu + yv + zw, xw + yu + zv, xv + yw + zu).$$

Obviously these two representations of the product are identical if $v = w$. Since g is a symmetric function of its arguments it is easy to see that they are identical in each of the following six cases: $v = w$, $v = u$, $w = u$, $x = y$, $x = z$, $y = z$. On the other hand if we assume that the two representations are identical we are led to one of the preceding six equalities. Thus we have the following theorem:*

THEOREM I. *If r, s, t have either of the two sets of values (r_1, s_1, t_1) and (r_2, s_2, t_2), where*

$$\begin{aligned} r_1 &= xu + yw + zv, & r_2 &= xu + yv + zw, \\ s_1 &= xv + yu + zw, & s_2 &= xw + yu + zv, \\ t_1 &= xw + yv + zu, & t_2 &= xv + yw + zu, \end{aligned} \tag{2}$$

then

$$g(x, y, z) \cdot g(u, v, w) = g(r, s, t).$$

In order that the two expressions $g(r, s, t)$ shall be non-identical it is necessary and sufficient that each of the two sets (x, y, z) and (u, v, w) shall consist of distinct members.

It may be observed that for each set of values (r, s, t) we have

$$r + s + t = (x + y + z)(u + v + w).$$

If a and b are both representable in the form g, then the product ab is representable in the same form, as is seen from the foregoing theorem. The question arises as to whether all the representations of ab are obtained by means of Theorem I from the representations of a and b. That this is to be answered in the negative follows from the simplest examples. Thus it is easy to show that 2 is represented in the form g in only one way, namely, $2 = g(1, 1, 0)$. From this and Theorem I we have $4 = g(2, 1, 1)$, the two sets (r, s, t) being equivalent in this case. But we have also $4 = g(1, 1, -1)$. That is, 4 is capable of a representation in the form g not obtainable by means of Theorem I from the representation of its proper factors.

* The result in this theorem is well known, as we have just pointed out. The remaining theorems in the paper are believed to be new.

From these two representations of 4 it follows that a number may be represented in two ways by the form g and yet these representations not result from writing the product of its factors in two ways in the form g by means of Theorem I. Two other examples illustrating this are afforded by the following relations: $20 = g(3, 1, 1) = g(7, 7, 6)$; $91 = g(6, 4, 3) = g(31, 30, 30)$.

We observe that if the numbers x, y, z, u, v, w in Theorem I are all non-negative then r, s, t are likewise non-negative. This leads us to consider the problem of the representation of numbers in the form g when the arguments are restricted to be non-negative. The fundamental theorem here is the following:

THEOREM II. *Every prime number p other than 3 is representable in one way and in only one way in the form*

$$p = g(x, y, z) \equiv x^3 + y^3 + z^3 - 3xyz, \tag{3}$$

where the arguments x, y, z are restricted to be non-negative.

In order to prove this let us seek to put p in the form

$$p = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

Since the numbers x, y, z are to be non-negative it is clear that this equation can be satisfied only when

$$x + y + z = p, \quad x^2 + y^2 + z^2 - xy - yz - zx = 1. \tag{4}$$

Without loss of generality we may assume that $x \geqq y \geqq z$, and this we do. Let us write

$$x = u + z, \quad y = v + z.$$

Then u, v, and $u - v$ are non-negative numbers. Equations (4) may now be written

$$3z + u + v = p, \quad u^2 - uv + v^2 = 1. \tag{5}$$

From the latter equation we have $(u - v)^2 + uv = 1$. From this it follows that $u = v = 1$ or $u = 1$, $v = 0$. From the first equation in (5) we see that the former set must be used when p is of the form $3k + 2$ and the latter when p is of the form $3k + 1$, in order that z shall be an integer. In either case u, v, z, and therefore x, y, z, are uniquely determined. Hence the theorem.

Now $g(2, 1, 0) = 9$. From this fact and Theorems I and II

it follows that every positive number is representable in the form $g(x, y, z)$ with non-negative arguments with the possible exception of those of the form $3t$, where t is not divisible by 3. Now, we have

$$g(x, y, z) = (x + y + z)\{(x + y + z)^2 - 3(xy + xz + yz)\}.$$

If the second member of this equation is divisible by 3, so is $x + y + z$, and therefore this second member is divisible by 9 (whatever signs x, y, z may have). Hence the form $g(x, y, z)$ does not contain any number $3t$ where t is an integer prime to 3. Thence we have the following theorem:

THEOREM III. *The positive integers which may be represented in the form $g(x, y, z)$ include all positive integers with the sole exception of those which are divisible by 3 but not by 9. In every case the arguments x, y, z in the representation may be chosen so as to be all non-negative.*

If x, y, z are allowed to be negative it is no longer true that primes are always uniquely represented in the form $g(x, y, z)$. Thus we have $7 = g(3, 2, 2) = g(2, -1, 0)$, $13 = g(5, 4, 4) = g(2, -2, 1)$. Then let us consider more generally the representation of a prime p in the form

$$p = g(x, y, z) = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz).$$

Writing $x = u + z$, $y = v + z$, we have

$$p = (3z + u + v)(u^2 - uv + v^2). \tag{6}$$

Now $4(u^2 - uv + v^2) = (u + v)^2 + 3(u - v)^2$, so that $u^2 - uv + v^2$ is not negative. Hence from (6) it follows that this expression has the value 1 or the value p. Therefore we have to examine the following two cases:

$$(a) \qquad u^2 - uv + v^2 = 1, \quad 3z + u + v = p;$$

$$(b) \qquad u^2 - uv + v^2 = p, \quad 3z + u + v = 1.$$

Now the equation $u^2 - uv + v^2 = 1$, or $(u + v)^2 + 3(u - v)^2 = 4$, has only the solutions obtained in the proof of Theorem II. Hence case (a) gives rise only to the representation by means of non-negative arguments x, y, z treated in Theorem II.

Let us next consider case (b). We have

$$4p = (u + v)^2 + 3(u - v)^2. \tag{7}$$

Since $p \neq 3$ it follows from this that $p \equiv 1 \bmod 3$, so that p is of the form $6n + 1$. Equation (7) has a solution for every prime p of the form $6n + 1$ such that $u + v \equiv 1 \bmod 3$.* Furthermore $u + v$ and $u - v$ are obviously both odd or both even, so that u and v are themselves integers. From the second equation in (b) it follows now that z, and hence x and y, are integers. Thus we have a representation of p in the desired form $g(x, y, z)$, one at least of the arguments x, y, z being obviously negative. Furthermore it is clear that this representation is unique provided that $4p$ has only one representation

$$4p = a^2 + 3b^2, \quad a > 0, \quad b > 0,$$

in which $a \equiv 1 \bmod 3$, since $u + v$ must have a value congruent to unity modulo 3 in order that z shall be an integer. This latter fact concerning $4p$ we shall now prove. Let $4p$ have the representation

$$4p = \alpha^2 + 3\beta^2, \quad \alpha > 0, \quad \beta > 0.$$

Then we have

$$(8)\quad 16p^2 = (a\alpha + 3b\beta)^2 + 3(a\beta - \alpha b)^2 = (a\alpha - 3b\beta)^2 + 3(a\beta + \alpha b)^2$$

and

$$(9)\qquad 4p(\alpha^2 - a^2) = 3(\alpha b + a\beta)(\alpha b - a\beta).$$

Hence p is a factor of $\alpha b + a\beta$ or of $\alpha b - a\beta$. Suppose that p is a factor of $\alpha b + a\beta$, the complementary factor being s. Then from (8) it follows that p is a factor of $a\alpha - 3b\beta$; let the complementary factor be t. Then from (8) we have

$$16 = t^2 + 3s^2;$$

whence $t = 4$, $s = 0$ or $t = s = 2$. If the former solution is taken, we find from (9) that $a = \alpha$ and hence that the two representations of $4p$ are identical. If we take the latter we have

$$\alpha b + a\beta = 2p, \quad a\alpha - 3b\beta = 2p;$$

whence it follows readily that

$$2\alpha = a + 3b.$$

* See Bachmann's Kreistheilung, pp. 138–141.

Since $a \equiv 1 \bmod 3$ it follows that $\alpha \equiv 2 \bmod 3$. In a similar way one may treat the case when $\alpha b - a\beta$ is divisible by p and with a similar result. Therefore $4p$ can be represented in the form $a^2 + 3b^2$ in only one way provided that a is restricted to be congruent to unity modulo 3.*

We are thus led to the following theorem:

THEOREM IV. *A prime number p of the form $6n + 1$ may be represented in one and in only one way in the form*

$$p = g(x, y, z) \equiv x^3 + y^3 + z^3 - 3xyz$$

where one at least of the arguments x, y, z is negative. No other prime number has such a representation. (Compare Theorem II.)

Let us next consider the representation of p^2 in the form $g(x, y, z)$, p being a prime number different from 3.† Writing $x = u + z$, $y = v + z$, we have

$$p^2 = (3z + u + v)(u^2 - uv + v^2).$$

Since $u^2 - uv + v^2$ cannot be negative it follows that there are three cases to be examined, namely:

$$(a) \qquad 3z + u + v = p^2, \qquad u^2 - uv + v^2 = 1;$$

$$(b) \qquad 3z + u + v = p, \qquad u^2 - uv + v^2 = p;$$

$$(c) \qquad 3z + u + v = 1, \qquad u^2 - uv + v^2 = p^2.$$

These may be treated by the methods already employed. We take up the cases in order.

The second equation in (a) has the two solutions $u = v = 1$; $u = 1, v = 0$, and no others (if we take $u \geqq v$, as we may without loss of generality). Since z must be integral it follows from the first equation in (a) that we must take $u = 1, v = 0$. We are thus led to the following conclusion:

There is a unique representation of p^2 $(p \neq 3)$ in the form $g(x, y, z)$ subject to the condition $x + y + z = p^2$.

In case (b) it is easy to show from the second equation that p is of the form $6n + 1$. Proceeding as in the proof of Theorem

* As a corollary of this argument we have the following result:
If p is a prime number of the form $6n + 1$ then $4p$ can be represented in two and in only two ways in the form $a^2 + 3b^2$, a and b being positive, and in one of these ways a is congruent to 1 and in the other a is congruent to 2 modulo 3.

† For the excluded case we have $9 = g(2, 1, 0)$.

IV, we find that there is a unique solution of equations (*b*) subject to the condition that z is integral. We thus conclude:

In order that p^2 $(p \neq 3)$ *shall be representable in the form* $g(x, y, z)$, *with the condition* $x + y + z = p$, *it is necessary and sufficient that* p *be of the form* $6n + 1$ *and this representation, when it exists, is unique.*

In case (*c*) the second equation has the obvious solution $u = v = p$. This solution will yield integral z only when p has the form $3k + 2$. The solution is unique for such p since it follows from the theory of binary quadratic forms that such a prime power p^2 can be represented in the form $u^2 - uv + v^2$ only when $u = v = p$ or $u = p, v = 0$, the latter solution giving z non-integral in the present case. If p is of the form $3k + 1$ then the second equation in (*c*) has the solution $u = p$, $v = 0$; this gives rise to integral z and hence to a representation of the kind sought. The representation in this case is not necessarily unique, since the second equation in (*c*) may have a second solution giving rise to integral z. We have the following result:

The prime power p^2 $(p \neq 3)$ *can be represented in the form* $g(x, y, z)$ *subject to the condition* $x + y + z = 1$.

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ON THE LINEAR CONTINUUM.

BY DR. ROBERT L. MOORE.

(Read before the American Mathematical Society, April 24, 1915.)

§ 1. *Introduction.*

IN the *Annals of Mathematics*, volume 16 (1915), pages 123–133, I proposed a set G of eight axioms for the linear continuum in terms of *point* and *limit*. Betweenness was defined,* and it was stated that the set G is categorical with respect to *point* and *the thus defined betweenness*.† In the present paper it is shown that, although this statement is true, nevertheless

* See Definition 3, loc. cit., p. 125.

† This statement, which is proved in the present paper, implies that if K is any statement in terms of point and betweenness, then either it follows from Axioms 1–8 and Definition 3 that K is true or it follows from Axioms 1–8 and Definition 3 that K is false.

G is not *absolutely* categorical,* that is to say it is not categorical with respect to *point* and *limit*, the undefined symbols in terms of which it is stated.

An absolutely categorical set is obtained if Axiom 5 is replaced by the following axiom.

AXIOM 5′. *If r_1 and r_2 are two mutually exclusive, non-complementary rays,† then every infinite set of points lying in $S - (r_1 + r_2)$ has at least one limit point.*

§ 2. *On the Non-Categoricity of the Set G.*

That G is not categorical is shown by the existence of the following examples E and E_5'. The letter K will be used to denote the statement that the point P is a *limit point* of the point set M whenever every segment containing P contains at least one point of M distinct from P.

E_5'. Let the space S be an ordinary linear continuum $(0 < x < 1)$ but interpret the statement that P is a *limit point* of M to mean that P is a limit point in the ordinary sense of a rational subset of M. Here Axioms 1–8 are satisfied‡ but statement K is false.

E. Let S be an ordinary linear continuum and let *limit*

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* For definition of categoricity (in the absolute sense) see O. Veblen, "A system of axioms for geometry," *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 343–384.

† If P is a point and S is the set of all points, and $S - P = S_P' + S_P''$, where S_P' and S_P'' are mutually exclusive connected point sets neither of which contains a limit point of the other one, then S_P' and S_P'' are called *rays*. If B is a point of the ray S_P' then the ray S_P' is denoted by PB. The ray S_P' is said to be complementary to (or the complement of) the ray S_P''.

‡ That Axiom 5 is satisfied in this example may be proved as follows. In this proof the phrase "limit point" (unitalicized) has its ordinary meaning while "*limit point*" (in italics) is to be interpreted as defined in E_5'.

Suppose that $S = K_1 + K_2$ where K_1 and K_2 are mutually exclusive point sets. There are two cases to be considered.

Case I. Suppose K_1 contains no rational subset. Then K_2 contains the set of all rational points. But every point of S is a limit point of this set. Hence every point of K_1 is a *limit point* of K_2.

Case II. Suppose $K_1 = R_1 + I_1$ and $K_2 = R_2 + I_2$, where R_1 and R_2 are composed entirely of rational points, while I_k $(k = 1, 2)$ either is vacuous or is composed entirely of irrational points. Suppose K_1 contains no limit point of R_2 and K_2 contains no limit point of R_1. Then one of the point sets I_1 and I_2 must contain a limit point of the other one. Suppose I_1 contains a limit point of I_2. Since every point of I_2 is a limit point of $R_1 + R_2$ but not of R_1, therefore every point of I_2 is a limit point of R_2. It follows that I_1 contains a limit point of R_2 and therefore K_1 contains a *limit point* of K_2.

point have its usual significance. In this example also Axioms 1–8 are all satisfied. But here the statement K is true.

From the existence of these two examples it is clear that neither K nor its contradictory is a consequence of Axioms 1–8. Hence this system of axioms is not absolutely categorical.

§ 3. *Consequences of Axioms 1–4, 6, 7.*

THEOREM A. *No point is a limit point of a finite set of points.*

Theorem A is a consequence of Axioms 2 and 3.

THEOREM B. *Every ray contains infinitely many points.*

Theorem B is a consequence of Theorem A and Theorem 1.*

THEOREM C. *Every ray contains an infinite set of points that has no limit point.*

Proof. By Axiom 7 there exists a countable set of points R such that every point either belongs to R or is a limit point of R. If the ray AB did not contain infinitely many points of R, then, by Theorem B, Axiom 2 and Theorem A, AB would contain a limit point of AB', its complement. But this is contrary to Definition 2. Hence AB and R contain infinitely many points in common. Let P_1, P_2, P_3 be the set of all such common points. By Theorem 11 there exists a point X_1 such that AP_1X_1. There exists K_1 such that AX_1K_1. By Theorem 2, AX_1 contains X_1K_1. But AX_1 is the same as AB. Thus AB contains X_1K_1. But, by Theorem 4, X_1P_1A. Therefore P_1 is on X_1A. Consequently P_1 is not on X_1K_1. Thus the ray X_1K_1 lies in AB but does not contain P_1. Similarly there exists a ray X_2K_2 lying in X_1K_1 (and therefore in AB) and not containing P_2. Continue this process, thus obtaining two sequences of points $X_1, X_2, \cdots$ and $K_1, K_2, \cdots$ such that AB contains X_nK_n, X_nK_n contains $X_{n+1}K_{n+1}$ and X_nK_n contains no point of the set $P_1, P_2, \cdots, P_n$. Suppose the infinite set of points $X_1, X_2, X_3, \cdots$ has a limit point X. The points $X_{n+1}, X_{n+2}, X_{n+3}, \cdots$ all lie on X_nK_n. Hence for every n, X lies on X_nK_n. Now A is not on X_nK_n. Hence it is not on X_nX. Therefore AX_nX is true for every n. Hence XA contains every X_n. But X_nA is the complement of X_nK_n, and therefore contains $P_1, P_2, P_3, \cdots, P_n$. Furthermore XA contains X_nA. Therefore XA contains all the points

* Arabic numerals are used for theorems and definitions contained in my paper "The linear continuum in terms of point and limit," loc. cit.

$P_1, P_2, \cdots, P_n$. But there exists a point Y such that AXY. The rays XY and XA are complementary. Therefore XY contains no P_n. But XY is a subset of AB. Therefore XY contains no point of R. Hence Y is not a limit point of R. Thus the supposition that the set of points $X_1, X_2, \cdots$ has a limit point leads to a contradiction.

§ 4. *Consequences of Axioms 1–4, 5′, 6, 7.*

THEOREM D.* *There do not exist three mutually exclusive rays.*

Theorem D is a consequence of Axiom 5′ and Theorem C.

THEOREM E. *If P is a limit point of M then every segment containing P contains at least one point of M distinct from P.*

Proof. Let AB denote a segment† containing P. There exist points C and D such that ABC and BAD. The ray BC is the complement of BA, while AD is the complement of AB. If the point X does not belong to the segment AB, then, by Theorems 14 and 4 and Definition 3, X is not common to the rays AB and BA. Hence if no point of M except P is in the segment AB then $M = M_1 + M_2$, where no point of M_1 except P is in AB and no point of M_2 is in BA. But P is in both AB and BA. Hence P is a limit point of neither M_1 nor M_2. Therefore, by Axiom 2, P is not a limit point of M. But this is contrary to hypothesis.

THEOREM F. *There exists a countable, everywhere dense‡ set of points.*

Theorem F is a consequence of Axiom 7 and Theorem E.

It follows§ from Theorems 4–14 and Theorem F that the set of Axioms 1–7‖ is categorical with respect to *point* and *betweenness* as defined in Definition 3.

THEOREM G. *If every segment containing P contains at least one point of M distinct from P then P is a limit point of M.*

Proof. Between S and the linear continuum $(0 < x < 1)$ there is a one-to-one reciprocal correspondence that *preserves*

* See Axiom 5, loc. cit., p. 126.

† The *segment* AB is the set of all points $[X]$ such that AXB.

‡ A set of points M is said to be everywhere dense if every segment contains a point of M.

§ See G. Cantor, "Zur Begründung der transfiniten Mengenlehre, I," *Mathematische Annalen*, vol. 46 (1895), p. 510.

‖ It is to be noted that Theorems E and F are both consequences of Axioms 1–7 as well as of Axioms 1–4, 5′, 6, 7.

order. It follows that M contains an infinite set of points $P_1, P_2, P_3, \cdots$ such that for every segment τ containing P there exists n such that $P_{n+1}, P_{n+2}, P_{n+3}, \cdots$ all lie in τ. It follows, with the help of Axiom 5′, that the set of points $P_1, P_2, P_3, \cdots$ has at least one limit point O. Suppose that O is distinct from P. Then there exist points A, B, and C in the order $APBOC$. There exists n such that $P_{n+1}, P_{n+2}, P_{n+3}, \cdots$ all lie in the segment AB. Hence not more than n points of the set $P_1 + P_2 + P_3 \cdots$ lie in the segment BC. Therefore, by Theorem E, Axiom 2 and Theorem A, O is not a limit point of $P_1 + P_2 + P_3 \cdots$. Thus the supposition that P is distinct from O leads to a contradiction. It follows that P is a limit point of $P_1 + P_2 + P_3 + \cdots$ and therefore of M.

§ 6. *Conclusion.*

THEOREM H. *The set of Axioms 1–4, 5′, 6, 7 is an absolutely categorical set of axioms for the linear continuum.*

Proof. It has been shown that this set of axioms is categorical with respect to point and betweenness as defined in Definition 3. But every statement in terms of *point* and *limit point* of a point set is,* in the presence of these axioms and Definition 3, equivalent to a statement in terms of *point* and *betweenness.* It follows that the set of Axioms 1–4, 5′, 6, 7 is categorical with respect to *point* and *limit point* of a point set.

That, in the set of Axioms 1–4, 5′, 6, 7, Axioms 2, 3, 4, 5′, and 6 are independent is shown by Examples E_2–E_6 of my paper in the *Annals.* That 1 and 7 are independent in this set is shown by the following examples, E_1 and E_7.

E_1. S is an ordinary linear continuum. The point P is a *limit point* of the point set M if and only if P is a limit point of M in the usual sense but M is not the set of all points.

E_7.† S is the set of all real number pairs (x, y) such that $0 < x < 1$ and $0 \leq y \leq 1$. The point (x_1, y_1) is a *limit point* of the point set M if, and only if, it is true that corresponding to each preassigned positive number ϵ there exists, in the set

* See Theorems E and G.

† The example E_7 was constructed with the assistance of an example given by Veblen in connection with his postulate of uniformity. Cf. O. Veblen, "Definition in terms of order alone in the linear continuum and in well-ordered sets," *Transactions of the American Mathematical Society,* vol. 6 (1905), p. 169.

M, a point (x, y), distinct from (x_1, y_1), such that 1) $|x - x_1| < \epsilon$, 2) $x - x_1 = 0$ and $|y - y_1| < \epsilon$, in case y_1 is distinct from 0 and from 1, 3) $x \leqq x_1$ if $y_1 = 0$, 4) $x \geqq x_1$ if $y_1 = 1$.

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A PROBLEM IN THE KINEMATICS OF A RIGID BODY.

BY PROFESSOR PETER FIELD.

THE problem of finding the acceleration of any point in a rigid body when the accelerations of three points are given, and incidentally of finding what is by this means determined regarding the velocities, has received but little attention. A theorem due to Burmeister solves the problem of finding the acceleration of any point in the plane of the three points whose accelerations are given. The theorem states: "If at four coplanar points P_1, P_2, P_3, P_4 the accelerations be drawn, their extremities lie in a plane and form a quadrilateral which is affine with the quadrilateral formed by the four points."

R. Mehmke* and J. Petersen† have considered the general case, but their results do not agree, owing to an oversight in Petersen's treatment. While their work is independent, the proof in both cases depends directly on the fact that when the distance between two points is constant the projections of their velocities on their joining line are equal and the projections of their accelerations on this line differ by $\omega^2 l$, l being the distance between the two points and ω the angular velocity of the line. The purpose of this paper is to show that the problem can be solved very simply by using the expressions for the accelerations which are ordinarily given in text books on mechanics, and by this method the kinematical meaning of the solution is also evident.

Let there be given the accelerations at three points. It is proposed to find what can be determined regarding the kinematical state of the body at the given instant. As the acceleration at any point in the plane of the three points can be

* Festschrift zur Feier des 50jährigen Bestehens der technischen Hochschule Darmstadt, page 77.

† Kinematik, page 46.

found by Burmeister's theorem, it is no restriction to assume that the points (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2) whose accelerations are given have at the given instant the coordinates (0, 0, 0), (1, 0, 0), (0, 1, 0).

The general formulas for the components $(\ddot{x}, \ddot{y}, \ddot{z})$ along the fixed axes of the acceleration of any point (x, y, z) of the moving body may be written*

$$\ddot{x} = \ddot{x}_0 + \omega_x(\omega_x x + \omega_y y + \omega_z z) - \omega^2 x + \dot{\omega}_y z - \dot{\omega}_z y,$$

$$\ddot{y} = \ddot{y}_0 + \omega_y(\omega_x x + \omega_y y + \omega_z z) - \omega^2 y + \dot{\omega}_z x - \dot{\omega}_x z,$$

$$\ddot{z} = \ddot{z}_0 + \omega_z(\omega_x x + \omega_y y + \omega_z z) - \omega^2 z + \dot{\omega}_x y - \dot{\omega}_y x.$$

In these equations ω is the angular velocity of the body, $\dot{\omega}$ the angular acceleration, and $(\ddot{x}_0, \ddot{y}_0, \ddot{z}_0)$ is the acceleration of that point in the body which at the given moment coincides with the origin of the axes of reference. These formulas applied to the points (x_1, y_1, z_1) and (x_2, y_2, z_2) give

$$\ddot{x}_1 = \ddot{x}_0 - (\omega_y^2 + \omega_z^2), \quad \ddot{y}_1 = \ddot{y}_0 + \omega_x\omega_y + \dot{\omega}_z, \quad \ddot{z}_1 = \ddot{z}_0 + \omega_x\omega_z - \dot{\omega}_y,$$

$$\ddot{x}_2 = \ddot{x}_0 + \omega_x\omega_y - \dot{\omega}_z, \quad \ddot{y}_2 = \ddot{y}_0 - (\omega_x^2 + \omega_z^2), \quad \ddot{z}_2 = \ddot{z}_0 + \omega_y\omega_z + \dot{\omega}_x.$$

These equations determine ω and $\dot{\omega}$ when the accelerations of the three points are given.

It is more convenient to say that one of the possible solutions gives p, q, r as the components of ω and l, m, n as the components of $\dot{\omega}$. Any other solution must satisfy the equations

$$q^2 + r^2 = \omega_y^2 + \omega_z^2, \quad pq + n = \omega_x\omega_y + \dot{\omega}_z, \quad pr - m = \omega_x\omega_z - \dot{\omega}_y,$$

$$pq - n = \omega_x\omega_y - \dot{\omega}_z, \quad p^2 + r^2 = \omega_x^2 + \omega_z^2, \quad qr + l = \omega_y\omega_z + \dot{\omega}_x.$$

It follows that the components of ω and $\dot{\omega}$ may be any one of the following:

I. (p, q, r), (l, m, n),

II. $(-p, -q, -r)$, (l, m, n),

III. $(p, q, -r)$, $(l + 2qr, m - 2pr, n)$,

IV. $(-p, -q, r)$, $(l + 2qr, m - 2pr, n)$.

This shows that the absolute value of ω is determined, but

* See for instance Ziwet and Field, Introduction to Analytical Mechanics, p. 107.

the direction of the axis of spin is not. The two axes lie in a plane perpendicular to the xy plane and they make equal angles with the xy plane. The two values of the angular acceleration have the same projection along the z axis but their projections on the xy plane differ by a vector which is perpendicular to the projection of ω on this plane and equal to $2r$ times this projection. [It might be more convenient to view the two values of $\dot{\omega}$ as having the components $l + qr \pm qr$, $m - pr \pm pr$, n.]

For I or II the components of the acceleration of any point (x, y, z) are

$$\begin{aligned} \ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x + (pq - n)y + (pr + m)z, \\ \ddot{y} &= \ddot{y}_0 + (pq + n)x - (p^2 + r^2)y + (qr - l)z, \\ \ddot{z} &= \ddot{z}_0 + (pr - m)x + (qr + l)y - (p^2 + q^2)z; \end{aligned} \tag{1}$$

for III or IV they are

$$\begin{aligned} \ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x + (pq - n)y + (m - 3pr)z, \\ \ddot{y} &= \ddot{y}_0 + (pq + n)x - (p^2 + r^2)y - (l + 3qr)z, \\ \ddot{z} &= \ddot{z}_0 + (pr - m)x + (l + qr)y - (p^2 + q^2)z. \end{aligned} \tag{2}$$

It is no restriction to take the axis of rotation in the yz plane, i. e., $p = 0$. In place of (1) and (2) we then have (1′) and (2′)

$$\begin{aligned} \ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x - ny + mz, \\ \ddot{y} &= \ddot{y}_0 + nx - r^2y + (qr - l)z, \\ \ddot{z} &= \ddot{z}_0 - mx + (qr + l)y - q^2z, \end{aligned} \tag{1′}$$

and

$$\begin{aligned} \ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x - ny + mz, \\ \ddot{y} &= \ddot{y}_0 + nx - r^2y - (l + 3qr)z, \\ \ddot{z} &= \ddot{z}_0 - mx + (l + qr)y - q^2z. \end{aligned} \tag{2′}$$

These equations show that there are two possible values for the acceleration of any point not in the xy plane. These values become equal if either q or r is equal to zero; i. e., if the axis of spin is either parallel or perpendicular to the plane of the three points. If neither q nor r is equal to zero, the center of acceleration is different for the two cases unless it should happen to lie in the xy plane.

Summary.—When the accelerations of three points in a rigid body are given, the acceleration of any point in the plane of the given points is determined uniquely. The acceleration of a point not in the plane of the given points is in general two valued. Moreover, there are in general four sets of values of ω and $\dot{\omega}$ which give the same values for the accelerations of the points in a given plane. For a given value of ω there can be determined the value of the spin for the line joining a given pair of points and hence the relative velocity of the two points can be found.

UNIVERSITY OF MICHIGAN,
September, 1915.

JULES HENRI POINCARÉ.

Enquête de "l'Enseignement Mathématique" sur la méthode de travail des mathématiciens. Publié par H. FEHR avec la collaboration de T. FLOURNOY et E. CLAPARÈDE. Deuxième édition conforme à la première suivie d'une *Note sur l'invention mathématique* par H. POINCARÉ. Paris, Gauthier-Villars, et Genève, George, 1912. 8vo. 8+137 pages. Price 5 francs.

Notice sur Henri Poincaré. Par E. LEBON. Paris, Hermann, 1913. 8vo. xlviii pages. Price 2 francs.

Savants du Jour: Henri Poincaré, Biographie, Bibliographie analytique des écrits. Seconde édition entièrement refondue. Par E. LEBON. Paris, Gauthier-Villars, 1912. Royal 8vo. 112 pages. Price 7 francs.

IT was in the latter part of 1900 that M. E. Maillet wrote as follows:* "Messieurs les Rédacteurs, Il y aurait, ce me semble, une tentative à faire, pour laquelle l'*Enseignement Mathématique* est à mon avis tout à fait désigné, et dont le succès pourrait rendre de bien grands services aux jeunes mathématiciens. Elle consisterait à ouvrir une sorte d'enquête auprès de savants connus; il s'agirait d'obtenir de chacun d'eux quelques renseignements personnels sur sa méthode de travail et de recherche, ses habitudes, l'hygiène générale qu'il juge la plus propre à faciliter son travail intellectuel, la manière de conduire le plus efficacement ses lectures et d'en tirer le meilleur parti, etc., etc. Je me borne ici à indiquer les grandes

* *L'Enseignement Mathématique*, 1901, tome 3, p. 58.

lignes, tout en reconnaissant que, si mon idée était mise en exécution, il y aurait lieu de pénétrer un peu plus dans le détail."

As an outcome of the letter of which this is the first paragraph, thirty questions were formulated and published,* and copies were distributed among mathematicians at the international congresses in Heidelberg and St. Louis; a certain number of copies were also addressed to savants of different countries. By 1905 a considerable body of material had been collected with reference to the questionnaire, and during the years 1905–1908† this material was published in suitable synoptic form. The whole was collected and first issued as a pamphlet in 1909.

The results of the pamphlet are deduced from documentary evidence of more than a hundred mathematicians who are, for the most part, living; about a score preferred to have their testimony published anonymously. In the inquiry America is represented by statements of such men as Professors Coolidge, E. W. Davis, Dickson, Emch, F. R. Moulton, Rietz, Snyder, J. W. Young, and Mr. Escott.

Here are some illustrations of the questionnaire and of the responses:

Question 1 (a): At what period of your recollection and under what circumstances did the taste for mathematics take possession of you? To this question 93 replies were received; 35 placed the period before 10 years of age, 43 from 11 to 15 years of age, 11 from 16 to 18 years, 3 from 19 to 20 years, and 1 at 26 years of age. M. Lecat reported, "At $3\frac{1}{2}$ my attention was strongly fixed on the idea of number;" Professor Dickson replied, "At the age of 5. . . . At 12 years of age I had decided to pursue mathematical study." The well-known facts concerning such infant mathematical prodigies as Clairaut, Gauss, Ampère, and Bertrand are also recalled. While Steiner showed early aptitude for oral calculation and astronomy, it was not till 18 years of age that he even learned to write.

Question 6: Have you sought to learn the genesis of the truths, discovered by you, to which you attach the greatest importance? *Question 7:* To what extent, in your opinion, do chance and inspiration play a part in mathematical discovery? Is this part always as large as it appears? *Question*

* *L'Enseignement Mathématique*, 1902, tome 4, pp. 208–211; tome 6, 1904, pp. 376–378, 481.

† *L'Enseignement Mathématique*. A complete list of references is given in tome 10, 1908, p. 172.

8(a): Have you sometimes remarked that discoveries or solutions in subjects entirely foreign to your researches of the moment have presented themselves to you, and that they corresponded to research formerly fruitless? *Question 8(b)*: Have you ever calculated or solved problems in a dream? Or have solutions and discoveries appeared to you in their completeness on waking up in the morning, when they were vainly sought in the evening or days previously? *Question 9:* Do you estimate that your principal discoveries have been the result of a voluntary endeavor directed in a definite way, or have they occurred to you spontaneously, so to speak?

The replies are summed up as follows (pages 47–48): "Mathematical discoveries—small or great, and whatever their content (new subjects of research, divination of methods or of lines to follow, presentiments of truths and of solutions not yet demonstrated, etc.)—are never born by spontaneous generation. They always presuppose a ground sown with preliminary knowledge and well prepared by work both conscious and subconscious. On the other hand every discovery by its very novelty and originality contrasts forcibly with such a statement and appears all the more surprising when it leaps forth unexpectedly from a very prolonged incubation. We learn, then, that according to the cases and the individuals it is sometimes the unlooked for character, and sometimes the dependence on previous work which strikes the author most when he reflects retrospectively. Whence so many varieties of estimate and equal truth of these two celebrated aphorisms, contradictory in appearance, but expressing the two aspects indissolubly bound together, although of relief often very unequal, of the same process: le génie, c'est l'inspiration; le génie, c'est une longe patience."

In connection with Question 8 (*b*), reference might have been made to the composition of Maria Agnesi's Instituzioni Analitiche (1748). "To this difficult task," H. J. Mozans writes,* "she devoted ten years of arduous and uninterrupted labor. And if we may credit her biographer, she consecrated the nights as well as the days to her herculean undertaking. For frequently, after working in vain on a difficult problem during the day, she was known to bound from her bed during the night while sound asleep, like a somnambulist, make her way through a long suite of rooms to her study, where she wrote

* Woman in Science, New York and London, 1913, pp. 144–145.

out the solution of the problem and then returned to bed. The following morning, on returning to her desk, she found to her great surprise, that while asleep she had fully solved the problem which had been the subject of her meditations during the day and of her dreams during the night."

Poincaré's " Note " on Mathematical Discovery is of extraordinary interest. It was delivered, in a conférence at the Institut général psychologique in May, 1908, and was first published in the Institut's *Bulletin* for June of the same year. A few months later it was included in the volume entitled Science et Méthode.* The opening sentences are as follows:

" The genesis of mathematical discovery is a problem which must inspire the psychologist with the keenest interest. For this is the process in which the human mind seems to borrow least from the exterior world, in which it acts, or appears to act, only by itself and on itself, so that by studying the process of geometric thought we may hope to arrive at what is most essential in the human mind.

" This has long been understood, and a few months ago a review called *l'Enseignement Mathématique* . . . instituted an inquiry into the habits of mind and methods of work of different mathematicians. I had outlined the principal features of this article when the results of the inquiry were published, so that I have hardly been able to make any use of them, and I will content myself with saying that the majority of the evidence confirms my conclusions."

The article deals mainly with just such topics as are suggested by questions quoted above. In the first of its five sections the author considers the following questions and answers: How does it happen that there are people who do not understand mathematics? How is error possible in mathematics? What is mathematical discovery? Poincaré points out that, especially in connection with discovery, intuition of mathematical order is fundamental. "A mathematical demonstration is not a simple juxtaposition of syllogisms; it consists of syllogisms *placed in a certain order*, and the order in which these elements are placed is much more important than the elements themselves. If I have the feeling, the intuition so to speak, of this order so that I can perceive the whole of

* Cf., for example, the English translation by Francis Maitland with a preface by B. Russell, London, 1914, pp. 46–63.

the argument at a glance, I need no longer be afraid of forgetting one of the elements; each of them will place itself naturally in the position prepared for it, without my having to make any effort of memory." "Discovery is discernment, selection," and selection is made by the intuition for order.

"But what I have said up to now," Poincaré remarks in commencing the second section, "is only what can be observed or inferred by reading the works of geometers, provided they are read with some reflection. It is time to penetrate further, and to see what happens in the very soul of the mathematician." And Poincaré recounts his recollections of how he wrote his first treatise on Fuchsian functions. Some of the underlying ideas are developed in the following section. Then there is a summing up in section four. "The result of all that precedes is to show that the unconscious ego or, as it is called, the subliminal ego, plays a most important part in mathematical discovery."

Poincaré considers as a first hypothesis: "The subliminal ego is in no way inferior to the conscious ego; it is not purely automatic; it is capable of discernment; it has tact and lightness of touch; it can select, and it can divine. More than that, it can divine better than the conscious ego, since it succeeds where the latter fails. In a word, is not the subliminal ego superior to the conscious ego?" Poincaré explains why he is loth to give an affirmative answer to this question. He prefers rather to consider: "Of the very large number of combinations which the subliminal ego blindly forms, almost all are without interest and without utility. But, for that very reason, they are without action on the esthetic sensibility; the consciousness will never know them. A few only are harmonious, and consequently at once useful and beautiful, and they will be capable of affecting the geometer's special sensibility I have been speaking of; which, once aroused, will direct our attention on them, and will thus give them the opportunity of becoming conscious." "The conscious ego is strictly limited." Limitations and characteristics of the subliminal ego are considered in the last section.

Such in barest outline are some of the thoughts.

M. Lebon's "Notice" is a reprint in pamphlet form of the introduction to the second edition of Poincaré's Hypothèses Cosmogoniques. The first half of the memoir is "Sur la vie

de Henri Poincaré," the latter, " Sur les travaux scientifiques de Henri Poincaré " and here M. Lebon tells us that his aim has only been to " signaler les beaux résultats des recherches originales de Henri Poincaré et les démonstrations rigoureuses qu'il donne, en insistant surtout sur les idées directrices de ses profondes études. D'une part, je souhaite avoir réussi à indiquer nettement comment son esprit fécond et universel est parvenu à élucider les théories obscures, à étendre le domaine des théories naissantes ou à en signaler les défauts. D'autre part, j'espère donner, à ceux qui ont en vue de faire progresser la Science, l'idée de lire avec attention tous ses écrits, qui leur montreront soit la meilleure marche à suivre, soit les points qu'il jugeait dignes d'être encore approfondis."

The first part of the " Notice " is especially interesting. As the topic is less dwelt upon in memorial sketches, a few quotations may be appropriately given in illustration of the whole.

" Henri Poincaré possessed, to a high degree, intuition of mathematical nature. At the Nancy lycée his comrades were struck by it. M. Paul Appell, his codisciple in Mathématiques Spéciales, affirms that he already had ' le don génial d'appercevoir intuitivement, avec le détail particulier de chaque question, l'idée générale dont elle procède, et la place qu'elle occupe dans l'ensemble.' From his first year at the lycée, Henri Poincaré had a method of working all his own. He had to force himself to sit at a study table, and neither noise nor conversation disturbed the working of his mind. To fix his thought on a subject there was no need for auxiliary material; it sufficed that a logical thread pervaded it, in order that it could not escape him. . . .

" Mathematician, philosopher, poet, artist, Henri Poincaré had to be also a great writer. His only aim was to express his thought with all his sincerity and to communicate to his readers his emotions and noblest enthusiasms. He wrote with a dash of the pen, for his ideas were of such a delicate nicety, his thoughts so excessively active, they almost always found immediately their perfect expression. . . .

" The style, infinitely supple and varied, is now the style of the savant, then that of the scholar or of the poet: it is also that of a writer truly French and of the line of the Montaignes, of the Molières, and of the Pascals. Elegant, simple, limpid, of great conciseness, this style abounds in amusing sallies, in an irony

occasionally cutting, but these sallies were aimed at ridiculous things, never at persons. It abounds also in pleasing and picturesque imagery set forth in ordinary language. But Henri Poincaré often happily rejuvenated the common expression by carrying to a conclusion the comparison that it implies or by imbuing a figure employed with originality, freshness, appealing power. . . .

"In scientific matters his only preoccupation was the search for truth. He concerned himself little with glory. He preferred that his name should not be given to any of his discoveries: to contemplate truth for an instant face to face was the only reward which appeared to him worthy of emulation. In acting thus he was certainly obedient to considerations of a lofty order, of an æsthetic order if one may so express it; but he was swayed by an impetuous sense of justice. To him the savants were all soldiers of one army. If in the common contest to ravish nature of her secrets some brilliant captains organize victory, it is nevertheless owing to the discipline, to the courage, to the endurance of the troops, that they win so completely. But, in the combat, how many brave soldiers fall 'sans laisser de noms, et après avoir utilement aidé à la victoire'"! . . .

"He had the happiness to unite his life to that of an intelligent companion, discreet and devoted, who embellished his existence and facilitated his tasks; for as M. Appell has expressed it, 'elle entourait son mari de l'atmosphère familiale, profondément unie et calme, qui seule permet les grands travaux de la pensée.' Henri Poincaré was the most tender and happy of fathers. He saw growing about him the children whom he loved profoundly and who recompensed him by their care always to have lurking about their lips the sweet smile of their affection and their joy. . . .

"As supreme consolation to those who loved him, Henri Poincaré has bequeathed to the centuries to come, with the example of a life as simple as it was beautiful and nobly completed, sa réconfortante pensée, sa foi en la grandeur, en la beauté de l'humanité. Son exemple et son oeuvre ont vaincu le néant.

> "'Recevant d'âge en âge une nouvelle vie,
> Ainsi s'en vout à Dieu les gloires d'autrefois;
> Ainsi le vaste écho de la voix du génie
> Devient du genre humain l'universelle voix. . . .'"

Although the new edition of Lebon's bio-bibliographical work is about a half larger than the old one* its general plan and appearance is the same. To the section on Biography have been added: (1) Darboux's reply to Poincaré's discourse at Darboux's Jubilee; (2) corrections and additions to the list of degrees, honorary titles, decorations, etc.; (3) titles of articles and works on Poincaré. The bibliographical part of Section II, on Mathematical Analysis, has been increased in size by 9 pages through the addition of details concerning new editions, reviews and new titles. In a similar way 5 pages have been added to Section III, on Analytical Mechanics and Celestial Mechanics, 6 pages to Section IV, on Mathematical Physics, and 2 pages to Section V, on Scientific Philosophy. History of Sciences is the title of Section VI, instead of "Necrology" in the old edition; the bibliography has been increased fourfold. Section VII is much the same. The total number of titles is 495, an increase of 59.

This work has been most admirably carried out and is beautifully printed and arranged. Errors and omissions at the date of publication, May 25, 1912, are probably very few. The error (page 85) in the page numbers (583 for 593) of Poincaré's note on non-euclidean geometry in the Traité de géométrie of Rouché and Camberousse† still persists. And Russian editions‡ of the conférence on "L'évolution des lois" (page 92) are overlooked.

As practically no reference has been made in this BULLETIN to recent memoirs on Poincaré, I append herewith a list of titles which I have met with, and which supplement Lebon's work. Those memoirs which are signed, are arranged alphabetically according to authors. It will be generally conceded, I believe, that Hadamard's memoir, on Poincaré as a mathematician, is of unsurpassed excellence.

R. d'Adhémar, (1) *Revue des questions scientifiques*, Brussels, vol. 22 (3), 1912, pp. 349–385; (2) Henri Poincaré ("Philosophes et Penseurs" série), Paris, Bloud et Gay, 1914. 12 mo. 64 pp.

* Reviewed by J. W. Young in this BULLETIN, October, 1910, vol. 17 (2), pp. 42–43.

† There are the same page numbers (581–593) for this note in the 8e éd. of the Traité, Paris, 1912.

‡ *Kagan's Bote* (formerly *Spaczinski's Bote*), Nr. 544, pp. 81–89 and Nr. 545, pp. 105–112, 1911. Also separately printed with E. Kohn's "Space and time from the standpoint of physics" under the title, in Russian, "Kohn and Poincaré: Space and time from the standpoint of physics." Odessa, *Mathesis*, 16mo., 81 pp.

—P. Appell, (1) *Revue du mois*, vol. 14, 1912, pp. 129–132; (2) *Annuaire Bureau des longitudes*, 1913, D. 14–18; (3) *Revue scientifique*, vol. 19, 1913, pp. 475–476, vol. 20, 1913, pp. 144–146.—K. Bajev, *Nouvelles de la société russe d'astronomie* (Russian), vol. 7, 1912, pp. 263–269.—R. Berthelot, Un romantisme utilitaire; étude sur le mouvement pragmatiste. Tome 1: Le pragmatisme chez Nietzsche et chez Poincaré. Paris, Alcan, 1911. (Deuxième partie; un pragmatisme scientifique. le pragmatisme fragmentaire et mitigé de Poincaré, pp. 195–416.)—Bigourdan, *Annuaire Bureau des longitudes*, 1913, D. 19–23.—E. Boutroux, *Revue de Paris*, vol. 20_1, pp. 673–702, vol. 20_2, pp. 77–91, 1913; also reprinted in pamphlet form, Coulomniers, 1913, 47 pp.—P. Boutroux, *Revue du mois*, vol. 15, 1913, pp. 155–183.—H. C. Brown, *Journal of Philosophy, Psychology and Scientific Methods*, New York, vol. 11, 1914, pp. 225–236.—L. Brunschvicg, "Poincaré le philosophe," *Revue de métaphysique et de morale*, vol. 21, 1913, pp. 585–616 (portrait of Poincaré in 1887.)—A. Buhl, *L'Enseignement Mathématique*, vol. 15, 1913, pp. 9–32 (portrait of Poincaré in his study).—R. H. Chassériaud, *Elektrotechnische Zeitschrift*, Berlin, vol. 33, 1912, p. 883.—J. Claretie, *Annuaire Bureau des longitudes*, 1913, D. 3–7.—Cornille, *Annuaire Bureau des longitudes*, 1913, D. 23–25.—S. Dickstein, *Wiadomości matematycne*, Warsaw, vol. 16, 1912, pp. 249–260 (portrait).—J. Echegazay, *Revista Soc. matem. española*, Madrid, vol. 2, 1912, pp. 33–39 (portrait).—G. Eichhorn, *Jahrbuch der drahtlosen Telegraphie*, Leipzig, 1912, vol. 6, pp. 109–113.—H. Fehr, *L'Enseignement Mathématique*, vol. 14, 1912, pp. 391–392.—M. Fouché, "La philosophie d'Henri Poincaré," *Bulletin de la Société Astronomique de France*, Paris, vol. 27, 1913, pp. 299–306. (Conférence faite à la séance du 2 Octobre, 1912.)—Galazine, *Bulletin de l'Acad. Impériale des Sciences de St. Pétersbourg* (Russian), 1912, pp. 819–820.—Gust'Hau, *Annuaire Bureau des longitudes*, 1913, D. 1–3.—J. Hadamard, (1) "Poincaré le mathématicien," *Revue de métaphysique et de morale*, vol. 21, 1913, pp. 617–658 (portrait of Poincaré in 1908); (2) "Henri Poincaré et le problème des trois corps," *Revue du mois*, vol. 16, 1913, pp. 385–418.—S. C. Haret, *Bulletin* (section scientifique) *Acad. roumaine*, Bucharest, vol. 1, 1912–13, pp. 50–65.—G. Humbert, *La Nature*, vol. 40_2, 1912, pp. 143–144 (1887 portrait, photo by Pirou).—P. E. B. Jourdain, *The Monist*, vol. 22, 1912, pp. 611–615.—A. Korn, *Sitzungsberichte d. mathem. Ges.*, Berlin, vol. 12, 1913, pp.

2–13 (portrait).—E. Lampe, *Naturwissenschaftliche Rundschau*, Braunschweig, vol. 27, 1912, pp. 476–479.—P. Langevin, "Poincaré le physicien," *Revue de métaphysique et de morale*, vol. 21, 1913, pp. 675–718; the same in *Revue du mois*, vol. 16, 1913, pp. 419–463.—A. Lebeuf, "Poincaré l'astronome," *Revue de métaphysique et de morale*, vol. 21, 1913, pp. 659–674.—Lippmann, (1) *Annuaire Bureau des longitudes*, 1913, D. 7–9; (2) *Comptes rendus de l'Acad. d. Sciences*, Paris, vol. 155, 1912, pp. 1280–1283.—A. E. H. Love, *Proceedings of the London Math. Society*, vol. 11 (2), 1913, pp. xli–xlviii.—P. Mansion, *Mathesis*, vol. 2 (4), 1912, pp. 233–238.—L. Margaillan, *Internationale Monatsschrift für Wissenschaft*, vol. 7, 1912, pp. 546–555.—F. Masson, *Revue scientifique*, vol. 18 (5), 1912, pp. 628–629.—C. Meyer, *Anales Soc. scientif. Argentina*, Buenos Ayres, vol. 74, 1912, pp. 125–147.—A. Mieli, *Rivista di filosofia*, vol. 5, 1913, pp. 44–48.—G. A. Miller, *Science*, New York, vol. 36 (2), 1912, pp. 425–429.—K. Mittenzwey, *Nord und Süd*, Berlin, vol. 147, 1913, pp. 53–58.—Morduchaj-Boltovskoj, *Bulletin de l'Université Impériale de Varsovie* (Russian), vol. 24, 1913, pp. 27–80.—L. T. More, "Poincaré and the Philosophy of Science," *The Nation*, New York, vol. 95, 1912, pp. 242–244.—F. R. Moulton, *Popular Astronomy*, vol. 20, 1912, pp. 621–634.—C. Nordmann, *Revue des deux mondes*, vol. 445, 1912, pp. 331–368.—L. Octavio de Toledo, *Revista soc. matem. espagñola*, Madrid, vol. 2, 1912, pp. 26–27.—P. Painlevé, (1) *Revue du mois*, 1912, pp. 132–134; (2) *Annuaire Bureau des longitudes*, 1913, D. 9–13.—E. Pascal, (1) *Giornali di matem.*, vol. 3 (3), 1912, pp. 303–309; (2) *Rendiconto Accad. d. Sc.*, Naples, vol. 18 (3), 1912, pp. 309–313.—E. Picard, *Annales scientif. de l'Ecole normale sup.*, Paris, vol. 30 (3), 1913, pp. 463–482; also printed separately.—G. Rageot, Les savants et la philosophie. Paris, Alcan, 1908. (Chapter 2: Le néo-criticisme d'un géomètre Henri Poincaré.)—L. Rougier, Henri Poincaré et la mort des vérités. Paris, *La phalange*, 1913, 22 pp.—G. Sarton, (1) *Ciel et terre*, Brussels, 1913, 25 pp. (portrait); (2) *Isis*, vol. 1, 1913, pp. 95–97 (portrait).—J. B. Shaw, "Poincaré as an Investigator," *Popular Science Monthly*, New York, vol. 82, 1913, pp. 209–224.—E. E. Slosson, "Major Prophets of to-day," *Independent*, vol. 71, 1911, pp. 729–741 (portrait); also in volume: Major Prophets of to-day, Boston, 1914, pp. 104–146.—W. B. Smith, *The Monist*, vol. 22, 1912, pp. 615–617.—C. Somigliana, *Atti della Reale Accademia della Scienze di*

Torino, vol. 49, 1914, pp. 45–54.—J. W. N. Sullivan, *Scientific American*, vol. 107, 1912, p. 78 (portrait).—I. Tschistiakov, *L'Enseignement Mathématique* (Russian), vol. 5, 1912, pp. 197–199.—G. Tzitzéica, *Gazeta matematica*, Bucharest, vol. 17, 1912, pp. 441–445.—O. Veblen, *Proceedings of the American Philosophical Society*, vol. 51, 1912, 9 pp.—V. Volterra, "Henri Poincaré: l'Oeuvre mathématique," *Revue du mois*, vol. 15, 1913, pp. 129–154.*—A. G. Webster, "Poincaré as a mathematical physicist," *Science*, vol. 38 (2), 1913, pp. 901–908.—H. Weyl, *Mathematisch-naturwissenschaftliche Blätter*, vol. 9, 1912, pp. 161–163.†

A few anonymous notes and sketches are to be found in: *Bulletin of the American Mathematical Society*, vol. 19 (2), 1912, p. 43.—*Nature*, vol. 90, 1912, pp. 353–356 (portrait plate supplement and autograph).—*Revue générale des Sciences*, vol. 23, 1912, p. 533.—*Revue scientifique*, vol. 17, 1912, p. 90.—*Sammlung und Mitteilungen und Protokolle der Math. Gesellschaft in Charkow* (Russian), vol. 13 (2), pp. 4–5.—*Supplemento ai Rendiconti del circolo matematico di Palermo*, vol. 8, 1913, pp. 13–32 (reprint of extracts from Lebon's work together with a facsimile of a Poincaré letter).—*Times*, London, July 18, 1912, p. 9, col. c.

In addition to the portraits listed above there is a full-page portrait and autograph in the *American Journal of Mathematics*, vol. 12, 1890; this is the 1887 portrait reproduced in the *Revue de métaphysique et de morale* and *La Nature*. An early portrait is reproduced in *Acta Mathematica*, 1882–1912, *Tables générale des Tomes 1–35*, Upsala et Stockholm, 1913, p. 164. In *Popular Science Monthly*, vol. 82, 1913, p. 412, there is a poor reproduction of the admirably life-like heliogravure frontispiece to Lebon's bio-bibliographical work.

Of the portraits above noted the best are those in the *American Journal etc.*, *L'Enseignement Mathématique*, *Revue de métaphysique etc.*, *Nature*, and Lebon's work.

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*The four memoirs by P. Boutroux, Hadamard, Langevin and Volterra, published in *La revue du mois*, have also been issued in book form with five pages of supplementary matter. Paris, Alcan, 1914. 12mo. 2 + 265 pp.

† It may be of interest to add a reference to a sketch of Poincaré prepared by Goursat. This consists mainly of extracts from the writings, mentioned above, of Lebon, Masson, P. Boutroux, V. Volterra, Humbert and Painlevé. The sketch is to be found in La vie et les travaux des savants modernes par A. Rebière; troisième édition revue et augmentée par E. Goursat. Paris [1913], pp. 188–203.

It is to be hoped that in the next edition of this work M. Lebon may be moved to give a list of papers and books which have been inspired by Henri Poincaré's suggestions and discoveries.

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SHORTER NOTICES.

First-Year Mathematics for Secondary Schools. By ERNST R. BRESLICH. Chicago, The University of Chicago Press. 1915. 344 pp.

ABOUT six hundred fifty years ago Roger Bacon gave voice to his feelings with respect to the teaching of mathematics, and this voice was in no respect uncertain nor was it at all lacking in emphasis. His words may be found in the Opus Majus, in the Opus Tertium, and in the manuscripts as yet unpublished of his De Communia Mathematicæ. In the last-mentioned work Bacon says that students are burdened with unnecessary difficulties to such a degree that they come to despise mathematics, whereas, if properly taught, the subject could be understood without any unreasonable expenditure of time; and that the first course in mathematics should not be designated as geometry, arithmetic, and so on, but as the elements of mathematics, a preliminary to the special branches.

What Bacon had to say on this phase of teaching was not new; others had said it before, and thousands have said it since, and after a fashion many have put the idea into practice. And so the effort of Mr. Breslich comes to the teaching profession as merely an ancient one clad in new guise. This does not in the least detract from the laudable nature of the effort, but it serves to give the work a kind of historical setting which assists us in judging of its novelty and its probable effect upon education.

The central idea of the work seems to be to select those features of secondary mathematics which are easily within the reach of beginners, postponing the consideration of the more difficult ones to a later period. As the author puts it, "The simpler principles are best suited for beginners, and may therefore be brought together in an introductory course."

In the pursuit of this idea the author proposes to treat of algebra and geometry at the same time, thus carrying out the ancient idea of fusion to which reference has been made above. He also proposes to consider those "subjects in which practical values are most clearly exhibited," to introduce a certain amount of trigonometry, to avoid "formalism in mode of presentation," and to give the student a "broader mathematical preparation."

With most of this ideal the educational world is generally in sympathy—perhaps with all of it except the mixing of algebra and geometry with no definite system. The following questions, however, will naturally arise in the minds of all who have to consider the book: Has the author carried out the plan successfully? That is, does the book meet the ideals which he has himself laid down? Given the average teacher, will the student, at the end of his work in the high school, be as well grounded in mathematics as he would have been if the work had been arranged on some other plan? Will he appreciate the subject as well or be as apt to continue his study of its higher branches?

In answer to the first question the reader is likely to hesitate before committing himself to an affirmative. He will find fully as much formalism in the early pages (for example, pages 5, 12, 20, 23) as he will find in any of the older types of algebra or geometry; he will find the commutative and associative laws given much earlier than the experience of teachers generally sanctions; he will find, for a work of this nature, an excessive number of definitions; he will find the rules of operation as formally stated as in the more common type of text-book; he will find the euclidean form of greatest common measure, with applications to numbers as large as those in the text-books of two generations ago; he will find such problems as the division of $2,400 into two parts having the ratio of 2 : 1 quite as he would find them in other books; he will find the simple made difficult in various cases, as in such products as $(a - b)c$, $(a + b)(a - b)$, $(a - b)(c + d)$, and $(a - b)^2$, and in the law of signs in multiplication as based on the "turning-tendency" idea; and he will find much the same type of problem that has come down to us from the past, as about a field that is twice as long as wide, and if it were 20 rd. longer and 24 rd. wider the area would be doubled. And when the serious inquirer has finished his reading of the book

he will have a feeling of doubt as to whether the plan of improving the first steps in mathematics has been as successful as he had hoped it would be.

And similarly as to the second question. Of course the book can be successfully taught; that is true of any book, provided the right teacher is available. But that a book with what seems to be a forced fusion of essentially different branches of a science, based solely upon the theory of ease of presentation, which theory does not seem to have been carried out—that such a book can be generally successful can hardly be expected.

It seems unfortunate that there should be in the book such statements as that "Pacioli in 1494 was the first to give rules for all processes of addition, subtraction, multiplication, and division" (page 20); that Tartaglia should be spoken of commonly as Fontana (facing page 158), when he himself preferred the former name, as the titles of his books prove; that Vieta or Viète should appear as Vièta (page 210); that the title of Fibonacci's work should be given as *Algebra et almuchabala*, when the manuscript actually begins "Incipit liber Abaci a leonardo filio Bonacij Pisano"; that the student should meet with the name Alkarismi (facing page 213) and with the inexcusable transliteration (unless with diacritical marks) of Al Hovarezmi a few pages later (page 255); that he should be told (page 20) that Diophantus lived about 250, and later (page 281) that he lived in the fourth century; and that numerous other slips of this kind should have been made in preparing the text.

The statement that "the coefficient of any factor in a term is the product of all the other factors of the term" will not seem very clear to a student who is told that 2 is the coefficient of x in the term $2x$. The treatment of negative numbers in Chapter XII will probably not seem to most teachers as clear as those to be found in our common algebras. The assertion that $a \times 0 = 0 \times a$ (page 195) will not seem warranted to those who may use the book, since the commutative law has not been referred to with respect to zero. Such problems as Example 31 on page 251, Example 10 on page 257, and numerous others of this type, will not lead teachers to feel that the author has broken away from the poorest type of inherited puzzles. These and criticisms like these will doubtless strike even the casual reader, and the causes for them will be regretted by all who wish success to any venture of this nature.

In the matter of skillful mathematical typography the book leaves more to be desired than is usually the case.

DAVID EUGENE SMITH.

Les Coordonnées intrinsèques, Théorie et Applications. Par L. BRAUDE. (Scientia, série physico-mathématique, no. 34.) Paris, Gauthier-Villars, 1914. 100 pp. Price 2 francs.

IN 1849 and 1850 William Whewell read two memoirs* on the intrinsic equation of a curve and its applications, before the Cambridge Philosophical Society. The opening paragraph of the first memoir is as follows:

"Mathematicians are aware how complex and intractable are generally the expressions for the lengths of curves referred to rectilinear coordinates, and also the determinations of their involutes and evolutes. It appears a natural reflexion to make, that this complexity arises in a considerable degree from the introduction into the investigation of the reference to the rectilinear coordinates (which are *extrinsic* lines); the properties of the curve lines with relation to these straight lines are something entirely extraneous, and additional with respect to the properties of the curves themselves, their involutes and evolutes; and the algebraical representation of the former class of properties may be very intricate and cumbrous, while there may exist some very simple and manageable expression of the properties of the curves when freed from these extraneous appendages. These considerations have led me to consider what would be the result if curves were expressed by means of a relation between two simple and *intrinsic* elements; the length of the curve and the angle through which it bends: and as this mode of expressing a curve much simplifies the solution of several problems, I shall state some of its consequences." He then considers the curve defined by the equation

$$s = f(\varphi), \tag{1}$$

points out that the radius of curvature follows at once from the relation

$$\rho = \frac{ds}{d\varphi} = F(\varphi), \tag{2}$$

* *Transactions of the Cambridge Philosophical Society*, vol. 8, part 5 (1849), pp. 659, 671; vol. 9, part 1 (1850), pp. 150, 156.

whence the equation of the evolutes,

$$s' = \frac{ds}{d\varphi} + C.$$

Similar relations result for successive evolutes and involutes.

Whewell then applies his discussion to the circle, equiangular spiral, cycloid, epicycloid, hypocycloid, "running pattern curves," catenary, and tractrix. He also derives some general properties, such as the derivation of the intrinsic equation of a curve when given its equation in rectangular coordinates and vice-versa, with applications to the parabola, ellipse, and semi-cubical parabola.

The second memoir contains further applications.

In connection with his first memoir Whewell remarks: "After writing this paper I found that Euler had, in the solution of a particular problem, expressed curves by means of an equation between the arc and the radius of curvature. This equation is, as is shown in the paper, the differential of my "intrinsic equation," and has an equally good right to the name. My equation being the integral of Euler's has, of course, one more arbitrary constant than his. There may very possibly be other modes of expressing curves which may be fitly described as "intrinsic equations" to the curves. I was not able to find any other name for the equation which I employed." No definite reference is given to Euler's works; but complete data of this nature, in connection with eight different papers, the first published in 1738 and the last in 1824, are given in Wölffing's bibliography.*

It may be remarked that while Whewell's memoirs† undoubtedly served to inspire the later elaborate developments of intrinsic geometry, and while French and Italians adopted Whewell's name "intrinsic," the same system of coordinates was also used, without special indication of its applications, by the German philosopher K. C. F. Krause (1802, 1804,

* E. Wölffing, "Bericht über den gegenwärtigen Stand der Lehre von den natürlichen Koordinaten" [in two dimensions]. *Bibliotheca Mathematica*, vol. 1(3) (1900), pp. 142–159. References are given to papers and books of nearly 90 different authors. M. Braude gives (p. 12) an incorrect reference to *Bibliotheca Mathematica*, vol. 2(3) (1901).

Intrinsic coordinates are discussed in the Encyklopädie der Math. Wissenschaften, Bd. III, 379 ff.; and Bd. III–3, pp. 34 ff., 84 ff., 198, etc.

† These memoirs were translated into German by A. Walter, Jahresbericht 1907 der k. k. ersten Staatsoberrealschule, Graz.

1835) and by A. Peters (Neue Curvenlehre, 1838), who is said to have introduced (Braude, page 9) the German term "natürliche Koordinaten" as opposed to Cartesian coordinates.

If equation (2) be looked upon as the polar equation of a curve, this curve is said to be the "radial curve" or radial of the original curve. These curves were so named by Robert Tucker who commenced their study in 1863. Four of his five papers on the subject are listed by Wölffing.* Tucker's definition was: "If from a point straight lines are drawn equal and parallel to the radii of curvature at successive points of a curve, their extremities will trace out the Radial Curve corresponding to the given curve."

The usual form of the intrinsic equation now employed is

$$f(s, \rho) = 0, \tag{3}$$

where ρ is the radius of curvature at a point of the curve defined by a given s. In a paper published in 1741 Euler discussed when an equation of this form defines an algebraic curve. If s and ρ be regarded as the rectangular coordinates of a point we get what Wölffing has called (1899) the Mannheim curve of the primitive curve, since it was in a memoir of 1859 that Mannheim remarked, among other properties, that the locus of the centers of curvature of the points of contact of this curve as it is rolled along a straight line is the curve defined by the intrinsic equation (3).

Again, if we consider the envelope of the line

$$x + y \tan \varphi - s = 0,$$

where s is the portion of the x-axis, measured from the origin, which is cut off by the given line, and φ the complement of the angle which the line makes with the x-axis, we get the tangential equation of the curve in the form of equation (1). Such a tangential equation has been studied at length by Casey and others. In a recent thesis by Koestlin,† however, this curve has been considered in its relation to the curve whose intrinsic equation is (1) and he has called it the "arcuïde" of the curve (1).

* The paper which Wölffing missed is in Mathematical Questions with their Solutions from the *Educational Times*, vol. 4, 1866, pp. 22–28.

† Koestlin, Ueber eine Deutung der Gleichung, die zwischen dem Bogen einer Kurve und dem Neigungswinkel der Tangente im Endpunkte des Bogens einer Kurve besteht. Tübingen, 1907.

The semi-intrinsic equations

$$f(s, x) = 0, \quad f(s, y) = 0$$

between the length of arc of a curve and the abscissa or the ordinate of the variable, have been discussed by several writers.

Sylvester and others have studied the analogous equation

$$f(s, r) = 0$$

between the arc and the radius vector.

Such are some of the principal forms of intrinsic equations used in the discussion of plane curves and their relations with one another.

The special importance of intrinsic coordinates was first brought out by Sophus Lie in his discussion of certain differential invariants in the theory of groups.* But this subject, as well as many applications of the coordinates to problems in differential and other geometry of curves and surfaces, is not considered in Braude's booklet. For these the reader must turn for guidance to the Encyklopädie or to the elaborate work of Cesàro: Lezioni di Geometria Intrinseca, first published at Naples in 1896.†

In the first 43 pages Braude gives "développements et méthodes" with numerous illustrative examples. Some subjects treated are: asymptotes, envelopes, successive evolutes, osculating curves, contact of higher orders, systems of curves and invariants of a system, and parallel curves. Special curves such as the epicycloids and hypocycloids, conics, logarithmic spiral, and causticoide are introduced in the examples.

The second section (pages 43–68) of the little book is taken up with a discussion of La Courbe de Mannheim. It is first introduced, as above, with a straight line as base; the generalization to a circle as base by Wieleitner and others is also developed with examples. And finally the circle is replaced as base by any plane curve. Another form of generalization

* Lie-Scheffers, Differentialgleichungen mit bekannten infinitesimalen Transformationen, Leipzig, 1891.

† M. Braude gives this date incorrectly (p. 6) as 1895. A German edition by G. Kowalewski: Vorlesungen über natürliche Geometrie, was published at Leipzig in 1901. Both editions were reviewed in this BULLETIN, vol. 9, Apr. 1903, pp. 349–357, by V. Snyder. The Italian edition was also reviewed by E. O. Lovitt, vol. 5 (March, 1899), pp. 303–306.

of Mannheim's curve was treated by that geometer himself and consists in seeking "the locus $M^{(n)}$ of the centers of curvature of order n"; $M^{(2)}$ would be the locus of the centers of curvature of order 2, that is, the evolute of $M^{(1)}$, the (first) Mannheim curve of the rolling curve. Then the general Mannheim curve $M^{(n)}$ (C, Γ) of a curve C, and with any plane curve Γ as base, is also discussed. Paragraphs on "intermediate evolutes," radials and the comparison of singularities, of corresponding radials and the curves of Mannheim, conclude the section. Again in the numerous examples many special curves are introduced. For example: (1) with a straight line as base, the Mannheim curve of a catenary is a parabola; (2) when the pseudo-cycloid or the logarithmic spiral is rolled on a curve Γ, we get as $M^{(4n)}$ (C, Γ) the curve Γ itself; $M^{(4n+3)}$ (C, Γ) is an involute of Γ.

The third section (pages 68–80) deals with the arcuïde. In particular there are sections on arcuïdes of algebraic curves, generation of the arcuïdes as glissettes, and on the generalization of the arcuïde by replacing the straight line by a curve as base (as in the case of Mannheim's curve).

The last section is entitled Les Roulettes, but without taking up too much space it is scarce possible to indicate further the nature of the discussion. Several theorems of Besant's Notes on roulettes and glissettes are recalled. M. Braude would have found still other interesting examples in the notable memoir written by the physicist J. Clark Maxwell at the age of seventeen: "On the theory of rolling curves."*

The work is highly accurate and takes account of recent developments of the subject, many of which are due to the author himself. The figures are admirably clear, although their face is sometimes hardly in keeping with the letter press. The style is concise but the numerous illustrative examples simplify the presentation of the theory which, in the very nature of the case, gives one the impression of being somewhat scrappy.

The volume is a very useful addition to the Scientia series and constitutes a pleasant introduction to Cesàro's great work.

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* *Proc. Royal Soc. Edinb.*, vol. 16, Feb., 1849, pp. 519–540; Collected Works, edited by Niven, 1890, vol. 1, pp. 4–29.

Leçons sur la Théorie des Nombres. Par A. CHÂTELET. Paris, Gauthier-Villars, 1913. x+156 pp.

THIS little volume consists, as the preface tells us, of the Peccot Foundation lectures in form substantially as delivered at the Collège de France in the second semester of the year 1911–12. Following the line of development given in his thesis, "Sur certains ensembles de tableaux et leur application à la théorie des nombres" (*Annales scientifiques de l'Ecole Normale Supérieure*, 1911), the author bases his exposition of the Dedekind theory of moduli and ideals largely upon the geometrical ideas of Minkowski and the so-called method of continued reduction of Hermite.

In the first chapter the algebraic foundation is laid by giving some necessary theorems concerning matrices and their relation to sets of forms, together with an all too brief account of Minkowski's theory of generalized distance. As in the thesis the term "*tableau*" is used throughout for a square matrix and the term matrix is employed to denote a rectangular array.

In the four following chapters the theory of Dedekind's moduli with its applications is studied in detail. If the sum and difference of two points be defined as in vector addition so that

$$(p_1, p_2, \cdots, p_n) \pm (q_1, q_2, \cdots, q_n) = (p_1 \pm q_1, p_2 \pm q_2, \cdots, p_n \pm q_n),$$

the definition of a modulus of points in agreement with Dedekind's definition of a modulus of numbers follows naturally. It is then easy to show that the coordinates of the points of the simplest modulus of dimension m in a space of dimension n are given by the matrix equation

$$||p_1, p_2, \cdots, p_n|| = ||x_1, x_2, \cdots, x_n|| \times A,$$

where the x's are integers and A is a matrix with m rows and n columns. The modulus is said to be "type" if A exists with the elements of each row coordinates of a point of the modulus such that every point of the modulus is given by the matrix equation. The matrix A is called a "base" of the modulus. The criterion for a type modulus is that it has only a finite number of points all of whose coordinates are less in absolute value than a given number. In general there are only a finite

number of points whose distance from a given point of the modulus is finite. The points of the modulus form a lattice work.

Passing to realms of algebraic numbers, the author interprets the n conjugate values of a number $\bar{\omega}$ in a realm $K(\omega)$ as the coordinates of a point in space, real or semi-real, of n dimensions. The elementary theorem which affirms that $\bar{\omega}$ may be expressed as a linear integral function with rational coefficients of the n powers of a primitive element ω of $K(\omega)$, gives the matrix equation

$$||\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n|| = ||a_0, a_1, \cdots, a_{n-1}|| \times \Omega,$$

where the a's are rational and Ω is the square root of the discriminant of ω. The numbers of the realm therefore give rise to a modulus R within which the integers of the realm form a type modulus C. The sub-modulus C is then according to Dedekind a multiple of R.

In the matrix equation for the conjugate values of an algebraic integer, the table Ω may be replaced by any other table

$$\Pi = R\Omega,$$

where the elements of R are rational and its determinant is not zero. It is then possible to determine π to a factor près in such a way as to establish an isomorphism between the numbers of the realm and the abelian system of tables

$$X_\omega = \pi[\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n]\pi^{-1}$$

with *rational* elements, where the bracket denotes a canonical table, i. e., a table in which the numbers $\bar{\omega}_i$ occupy the principal diagonal and all other elements are zero.

Similarly, when the integers of a realm are studied, it is easily shown that the integers of any ideal of the realm form a sub-modulus A of C which is again type and which is given by the matrix equation

$$||\beta_1, \beta_2, \cdots, \beta_n|| = ||x_1, x_2, \cdots, x_n|| \times PT,$$

where the x's are integers, T is a base of the realm, and P is a table with rational elements. The table PT will be "a base relative to an ideal" if, and only if, the elements of the table

$$(PT)[\alpha_1, \alpha_2, \cdots, \alpha_n](PT)^{-1}$$

are integers for every integer α of the realm.

In the two remaining chapters, devoted to continual reduction and the theorems of Minkowski and to the reduction of a base of a realm, geometrical ideas are brought even more clearly into evidence. In a modulus of points of dimension n in space of n dimensions, the points give rise to a totality of tables each of which is formed of the coordinates of n linearly independent points. Among the tables of such a totality it is natural that we should seek for the simplest. But what shall be the criterion? Geometrical intuition suggests at once that we choose those tables whose points lie as near as possible to the origin. But here a difficulty confronts us, for it may be that several points are equally distant from the origin and we cannot choose between them. These difficulties are overcome if for the ordinary distance we substitute Minkowski's "span," which is defined as the maximum of the absolute values of the differences between the coordinates of the two points. Thus the span $S(OA)$ of a point A from the origin is the maximum of the absolute values of its coordinates. Further, when two points have the same span with respect to the origin, we may distinguish between the ranks of their spans if we define "rank of span" to be the number reading from left to right of the maximum coordinate. For example, the two points A_1 (1, 5, 5) and A_2 (2, 3, 5) have the same span but the rank of A_1 is 2 while the rank of A_2 is 3.

A simplest or "reduced table" is one whose rows are the coordinates of n linearly independent points $A_i(p_1^{(i)}, p_2^{(i)}, \cdots, p_n^{(i)})$ which satisfy the following conditions:

(1) $$S(OA_1) \leqq S(OA_2) \leqq \cdots \leqq S(OA_n),$$

(2) $$p_1^i > 0,$$

(3) If $S(OA_i) = S(OA_{i+1})$, rank $S(OA_i) \leqq S(OA_{i+1})$.

For a given modulus the number of reduced tables is finite, as is geometrically evident. Nevertheless, it is possible to subject the modulus to a dilatation with continuous parameters so that for each of an infinite number of systems of values of the parameters a finite number of reduced tables exist. The importance of this method of introducing continuous parameters, which is Hermite's method of continuous reduction, lies in the fact that in a very important case, namely the case where the table is real and consequently associated with a decomposable form, the table is given only to a dilatation près.

When $n = 2$ and the elements of the table are real the series of reduced tables may be arranged, one for each interval, as a single parameter λ ranges both ways from a given value λ_1 defining a series of intervals such that

$$0 < \cdots < \lambda_{-2} < \lambda_{-1} < \lambda_1 < \lambda_2 < \cdots < \infty,$$
$$(\lim \lambda_i = \infty, \lim \lambda_{-i} = 0).$$

For the general case the reduction is accomplished by means of the fundamental inequalities of Minkowski. The two fundamental problems of finding the units of a realm and of separating the ideals of a realm into classes of equivalent ideals are made to depend upon the reduction of a base.

Three notes, the first on the application of the theory of moduli to periods of functions, the second a study of the realm $K(\sqrt{82})$, and the third a brief account of congruences with respect to an ideal and with respect to the norm of an ideal, occupy the last twenty-four pages of the book.

M. Châtelet modestly disclaims for the book any originality so far as material is concerned. But if it contains no hitherto unpublished results, the treatment is sufficiently novel to make the book a noteworthy contribution to the literature of algebraic numbers. The bringing together of the algebraic analysis of Hermite and the geometrical researches of Minkowski as aids to the development of the brilliant conceptions of Kummer and Dedekind is an achievement for which the mathematical world owes much to the author.

The book as a whole is well written, though at times it is brief almost to the point of obscurity. For the ordinary reader its value would have been greatly enhanced by additional concrete illustrations, and by even a few figures similar to those which illuminate Minkowski's Diophantische Approximationen.

E. B. SKINNER.

A Treatise on the Analytic Geometry of Three Dimensions. By GEORGE SALMON. Fifth edition, volume 2, edited by R. A. P. ROGERS. London, Longmans, Green and Company, 1915. xvi+334 pp.

THE second volume of the fifth edition of the Treatise begins with families of surfaces, which was Chapter XIII of the fourth edition. The numbering of the chapters has been retained.

The first section has not been changed; the second, on line geometry, has been enlarged from six to forty pages, and made into a new chapter, XIII*a*. It first considers the linear complex, derives the canonical equation, and gives a few important examples. After defining a quadratic complex, the treatment concerns differential properties, including a discussion of singular complexes, singular surface, etc., and then applies the results to the quadratic complex. The singular surface is obtained and a number of properties derived, including the existence of the six special congruences. Then follow a few general properties of congruences of lines, focal surfaces, limiting surface, developables, normal and isotropic congruences. Too many interesting properties are crowded into fine print and given as examples. The part on ruled surfaces is hardly changed.

Section III of Chapter XIII of the fourth edition is also made into a new chapter, XIII*b*. The change consists of an addition of fourteen pages on Lamé's curvilinear coordinates, and Darboux's theorem concerning the intersections of orthogonal surfaces; normal congruences of curves and cyclic systems close the chapter.

In Chapter XIV some bibliographical references are inserted at the beginning. Only one short article on inversion is added. The chapter on the cubic surface is amplified by various minor additions and numerous examples, but the general treatment is not changed.

A number of additions to the theory of the quartic surface have been made; much modern literature has been cited, but by no means all of the important memoirs have been given.

The text of the former edition has been so faithfully reproduced that a serious error concerning plane sections of a ruled surface, passing through a straight line director, is found in the new book (page 203). Two pages are given to Steiner's quartic, one to mapping a quartic surface with a double line on a plane. A short explanation of the complex surface of Plücker, and a fuller discussion of the cones of Kummer connected with the general cyclide, a few pages on the Weddle surface giving an incomplete summary of the later development of this theory, an amplified description of the symmetroid, and the explanation of the Kummer surface rewritten constitute the changes in this chapter.

In the section on the transformations of surfaces, an intro-

duction to the general theory of Cremona transformations is inserted, but so brief and containing so many statements without proof that its value is much less than could justly be expected. A fairly full discussion of the (2, 2) case is given, also an outline of the (2, 3) and (2, 4) cases, but no claim is made for completeness. A brief statement concerning Segre's work on the resolution of singularities by means of quadratic transformations is misleading. The problem was by no means completely solved by Segre.

With the tremendous amount of new material that has been contributed to this field during the last few decades it is difficult to develop a systematic theory without expanding into several volumes. The editor has endeavored to present these new phases in as near the same degree of completeness as was adopted by the author toward the corresponding field when the last preceding edition was written. This has, on the whole, been accomplished. The same style has been followed and a book has been produced, which "forms, it is hoped, a concise and comprehensive survey of tri-dimensional euclidean geometry, both algebraic and differential."

VIRGIL SNYDER.

Hermann Grassmanns gesammelte mathematische und physikalische Werke. Herausgegeben von FRIEDRICH ENGEL. Bd. 3: Teil 1, *Theorie der Ebbe und Flut und Abhandlungen zur mathematischen Physik*, 353 pp., 1911 (herausgegeben von JUSTUS GRASSMANN und FRIEDRICH ENGEL); Teil 2, *Grassmanns Leben*, xiii + 400 pp., 1911 (geschildert von FRIEDRICH ENGEL). Leipzig, B. G. Teubner.

GRASSMANN'S works on mathematics and physics are finished after some twenty years of editorial labor.

Although Grassmann's style was such as to repel readers from the two Ausdehnungslehren, those works have slowly penetrated into the mathematical consciousness of at least a few persons, but the first part of Volume III of his works contains material that is now printed for the first time and, though written fifty or seventy-five years ago, has only now an opportunity to be valuable to the world at large. The world of science has meantime moved far on, and about the only interest in this early work must be historical, not for the history of science, merely for the history of Grassmann.

The most striking thing about the memoir Ebbe und Flut,

which was finished by April, 1840, is the great development which Grassmann had already given to his system of geometric analysis. Here we find not only vector addition and subtraction, the scalar and vector products, differentiation by a scalar, and an incisive treatment of mechanical theory by them, but we find also divergence, the linear vector function, elliptic harmonic motion and the integration of vector differential equations, and rotary operators. The work further shows that at thirty years of age Grassmann, with practically no help, had made himself master of Lagrange and, to a considerable extent at least, of Laplace, so that he could improve on both.

The mathematical physics in this volume is from the Nachlass, and is chiefly analytical optics, some of it more like Gibbs's treatment in his lectures than anything we have seen in print.

For composing the life of Grassmann, Engel had at his disposal a number of documents written by Grassmann himself or by members of his family. It was therefore possible to begin the sketch away back in 1634 (!) and to offer a detailed treatment of the early years of this very busy genius. It is interesting to see how systematically he went to work at the university to broaden his knowledge and render it fundamental in a number of subjects. We may therefore be able a little to understand how Grassmann, who started as a theologian, could have been inventing his analysis, writing on physics, composing class texts for the study of German, Latin, and mathematics, editing a political paper, a missionary paper, investigating phonetic laws, writing a dictionary to the Rig-Veda, publishing a translation of the Rig-Veda in verse, and harmonizing folk songs in three voices, at various times of his life and frequently at the same time—in addition to carrying on successfully his regular work as a teacher and administrator and bringing up nine of his eleven children.

Engel's life of Grassmann is written in a sound critical spirit, there is neither laudation nor condemnation of its subject, merely a connected and sympathetic history of him, from which the reader may get instruction and interest and inspiration. Few biographers in science have had a harder task, for few scientists have had a wider range of activity than Grassmann. We should all admit our deep obligations to Engel as editor and biographer.

E. B. WILSON.

NOTES.

THE twenty-second annual meeting of the American Mathematical Society will be held in New York City on Monday and Tuesday, December 27–28, 1915. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary by December 11. Abstracts intended to be printed in advance of the meeting should be sent in by December 4.

THE thirty-sixth regular meeting of the Chicago Section, being the fifth western meeting of the Society, will be held at Columbus, Ohio, on Thursday, Friday, and Saturday, December 30–31 and January 1, in affiliation with the American association for the advancement of science. The first session will be a joint meeting with Section A of the Association, at which Professor H. S. WHITE will deliver his retiring address as vice-president of Section A, on "Poncelet polygons," and Professor E. J. WILCZYNSKI will deliver his retiring address as chairman of the Chicago Section. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary of the Chicago Section by December 4.

A NEW edition of the List of Officers and Members of the Society is now in preparation and will be issued in January. Blanks for furnishing information have been sent to the members. A prompt response will contribute materially to the correctness and completeness of the List.

THE sixty-eighth meeting of the American association for the advancement of science will be held at Columbus, Ohio, December 27 to January 1, under the presidency of Professor W. W. CAMPBELL, of Lick Observatory. Professor A. O. LEUSCHNER is vice-president and Professor F. R. MOULTON secretary of Section A.

THE concluding (October) number of volume 16 of the *Transactions of the American Mathematical Society* contains the following papers: "A type of singular points for a transformation of three variables," by W. V. LOVITT; "The re-

duction of multiple L-integrals of separated functions to iterated L-integrals," by J. K. LAMOND; "Independent generators of a group of finite order," by G. A. MILLER; "On the zeros of the function $P(x)$, complementary to the incomplete gamma function," by C. N. HASKINS; "Group properties of the residue classes of certain Kronecker modular systems and some related generalizations in number theory," by E. KIRCHER; "Sur l'intégrale de Lebesgue," by C. DE LA VALLÉE POUSSIN; "A new development of the theory of algebraic numbers," by G. E. WAHLIN; "Ruled surfaces whose flecnode curves have plane branches," by A. F. CARPENTER.

THE closing (October) number of volume 37 of the *American Journal of Mathematics* contains the following papers: "Geometrical and invariantive theory of quartic curves modulo 2," by L. E. DICKSON; "On the solutions of linear non-homogeneous partial differential equations," by L. L. STEIMLEY; "A method in the calculus of variations," by R. B. ROBBINS; "On the conformal geometry of analytic arcs," by G. A. PFEIFFER; "The non-homogeneous differential equation of parabolic type," by G. C. EVANS; "On properties of the solutions of linear q-difference equations with entire function coefficients," by T. E. MASON; "On rational sextic surfaces having a nodal curve of order 9," by C. H. SISAM.

THE annual list of American doctorates published in *Science* presents, for the academic year 1914–1915, 556 names, of which 309 are credited to the sciences. The following 25 successful candidates offered mathematics as major subject (the titles of the theses are appended): A. A. BENNETT, Princeton, "An algebraic treatment of the theorem of closure"; J. W. CAMPBELL, Chicago, "Periodic solutions of the problem of three bodies in three dimensions"; C. R. DINES, Chicago, "Functions of positive type and related topics in general analysis"; C. H. FORSYTH, Michigan, "Vital and monetary losses in the United States due to preventable deaths"; M. G. GABA, Chicago, "A set of postulates for general projective geometry of n dimensions"; OLIVE C. HAZLETT, Chicago, "On the classification and invariantive characterization of nilpotent algebras"; H. B. HEDRICK, Yale, "Some principles and processes in the construction of mathematical tables";

L. A. HOPKINS, Chicago, "On the theory of the motion of the small planets with a periodic orbit for the Hilda type"; H. R. KINGSTON, Chicago, "Metric properties of nets of plane curves"; W. V. LOVITT, Chicago, "A type of singular points for a transformation of three variables"; W. E. MILNE, Harvard, "On the degree of convergence of Birkhoff's series"; G. A. PFEIFFER, Columbia, "On the conformal geometry of analytic arcs"; V. C. POOR, Chicago, "A certain type of exact solution of the equations of motion of a viscous liquid"; H. F. PRICE, Pennsylvania, "Fundamental regions for certain finite groups in two complex variables"; L. J. REED, Pennsylvania, "Some fundamental systems of formal modular invariants and covariants"; P. R. RIDER, Yale, "An extension of Bliss's form of the problem of the calculus of variations, with applications to the generalization of angle"; J. ROSENBAUM, Cornell, "On mixed linear integral equations over a two-dimensional region"; G. RUTLEDGE, Illinois, "The number of abelian subgroups of groups whose orders are the powers of primes"; CAROLINE E. SEELY, Columbia, "Certain non-linear integral equations"; C. P. SOUSLEY, Johns Hopkins, "Invariants and covariants of the Cremona hexahedral form of the cubic surface"; EULA A. WEEKS, Missouri, "A symmetrical generalization of the theory of functions"; C. J. WEST, Cornell, "On certain formulas for representing statistical data"; C. E. WILDER, Harvard, "Problems in the theory of ordinary linear differential equations with auxiliary conditions at more than two points"; F. B. WILEY, Chicago, "Proof of the finiteness of the modular covariants of a system of binary forms and cogredient points"; L. T. WILSON, Harvard, "Conformal transformation of curvilinear angles."

THE United States Bureau of education has recently issued the following two bulletins: No. 35, on "Mathematics in the lower and middle commercial and industrial schools of various countries represented in the International commission on the teaching of mathematics," prepared by Dr. E. H. TAYLOR with the editorial cooperation of the American members of the Commission; No. 39, on "The training of elementary school teachers in mathematics in the countries represented by the Commission," by Dr. I. L. KANDEL. These bulletins may be obtained by teachers of mathematics on application to the Bureau.

THE English translation, by Professor E. R. HEDRICK, of the revised edition of the second volume of Goursat's Mathematical Analysis is expected to appear in January, 1916.

THE Joint committee on standards for graphic presentation, composed of representatives of a number of national organizations, has issued a preliminary report as a basis for suggestions to the committee. Copies can be obtained from the American society of mechanical engineers, 29 West 39th Street, New York City, at 10 cents each.

AT the University of Kansas, Dr. C. H. ASHTON has been promoted from an associate professorship to a full professorship of mathematics.

DR. E. C. COLPITTS, of the State College of Washington, has been promoted from an assistant professorship to an associate professorship of mathematics.

MISS GERTRUDE I. MCCAIN has been appointed professor of mathematics in the Western College for Women, Oxford, Ohio.

PROFESSOR E. H. JONES, of Daniel Baker College, has been appointed associate professor of mathematics in the Southern Methodist University.

AT Williams College Dr. H. L. AGARD has been promoted to an assistant professorship of mathematics.

DR. S. D. KILLAM, of the University of Alberta, has been promoted to an assistant professorship of applied mathematics.

THE following appointments to instructorships in mathematics are announced: Dr. M. G. GABA, at Cornell University; Dr. L. M. KELLS, at the College of the City of New York; Mr. J. S. MIKESH, at the University of Minnesota; Dr. P. R. RIDER, at the Sheffield Scientific School.

BOOK catalogues: The Macmillan Company, 66 Fifth Avenue, New York City, catalogue of mathematical and

astronomical books, 1915–1916.—George Gregory, 5 Argyle Street, Bath, England, catalogue no. 240–241, higher mathematics, 38 items.—Galloway and Porter, Cambridge, England, recent mathematical acquisitions, 16 items.—Arthur H. Clark Company, Caxton Building, Cleveland, Ohio, early and rare books, 27 items.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARCHIMEDES. Opera omnia cum commentariis Eutocii. Iterum J. L. Heiberg. Vol. III. Leipzig, Teubner, 1915. 8vo. 98+448 pp. M. 9.00

AUTONNE (L.). Sur les matrices hypo-hermitiennes et sur les matrices unitaires. Paris, Gauthier-Villars, 1915. 4+78 pp. Fr.5.00

BACHMANN (P.). See ENCYCLOPÉDIE.

BIRKELAND (K.) et SKOLEM (T.). Une méthode énumérative de la géométrie. Christiania (Vid. Selsk. Skrift.), 1915. Gr. 8vo. 61 pp.

CAHEN (E.). See ENCYCLOPÉDIE.

CANTOR (G.). Contributions to the founding of the theory of transfinite numbers. Translated by P. E. B. Jourdain. Chicago and London, Open Court, 1915. 8vo. 10+211 pp. Cloth. $1.25

DEMORGAN (A.). A budget of paradoxes. 2d edition, edited by D. E. Smith. 2 volumes. Chicago, Open Court, 1915. 8vo. 8+402+387 pp. Cloth. $7.00

DICKSON (L. E.). Algebraic invariants. (Mathematical monograph series, No. 14.) New York, Wiley, 1915. 8vo. $1.25

DINGELDEY (F.). See SALMON (G.)

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome I, volume 3, fascicule 5: Propositions transcendantes de la théorie des nombres, par P. Bachmann, J. Hadamard et E. Maillet; Théorie des corps de nombres algébriques, par D. Hilbert et H. Vogt; Multiplication complexe, par H. Weber et E. Cahen. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 385–480. M. 3.00

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 3, Heft 4: H. Liebmann, Berührungstransformationen; Geometrische Theorie der Differentialgleichungen. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 441–539. M. 3.20

ENGEL (F.). Zur Differentialgeometrie der komplexen analytischen Flächen. Strassburg, 1914. 8vo. 37 pp.

FIEDLER (W.). See SALMON (G.).

GOLDENRING (R.). Die elementargeometrischen Konstruktionen des regelmässigen Siebzehnecks. Eine historisch-kritische Darstellung. Leipzig, Teubner, 1915. Gr. 8vo. 7+69 pp. Geh. M. 2.80

GOURSAT (E.). Cours d'analyse mathématique. 2e édition, entièrement refondu. Tome 3: Intégrales infiniment voisines. Equations aux dérivées partielles du second ordre. Equations intégrales. Calcul des variations. Paris, Gauthier-Villars, 1915. Gr. 8vo. 667 pp. Fr. 20.00

GUIMARAES (R.). Sur la vie et l'œuvre de Pedro Nuñes. Coïmbre, Imprimerie de l'Université, 1915. 8vo. 87 pp.

GUMPPENBERG (H. v.). Beweis des grossen Fermatschen Satzes für alle ungeraden Exponenten. München, 1915. Gr. 8vo. 124 pp. M. 2.80

HADAMARD (J.). See ENCYCLOPÉDIE.

HEIBERG (J. L.). See ARCHIMEDES.

HILBERT (D.). See ENCYCLOPÉDIE.

JORDI (E.). Ueber Reihenentwicklungen nach Quadraten und Produkten von Besselschen Funktionen. Bern, 1913. 8vo. 111 pp.

JOURDAIN (P. E. B.). See CANTOR (G.).

KAYE (G. R.). Indian mathematics. Calcutta and Simla, Thacker, Spink and Co., 1915. 73 pp.

KLOOSTERBOER (G. W.). Coördinatentafel. Sinus- en cosinustafels ter berekening von rechthoekige coördinaten. Deventer, 1914. Fol. Geb. M. 4.80

LANDIS (E. H.). See RICHARDSON (R. P.).

LIEBMANN (H.). See ENCYKLOPÄDIE.

LORENTZ (H. A.). Lehrbuch der Differential- und Integralrechnung nebst Einführung in andere Teile der Mathematik, mit besonderer Berücksichtigung der Bedürfnisse der Studierenden der Naturwissenschaften. Uebersetzt von G. C. Schmidt. 3te Auflage. Leipzig, 1915. 8vo. 7+602 pp. M. 14.00

MAILLET (E.). See ENCYCLOPÉDIE.

NALLI (P.). Esposizione e confronto critico delle diverse definizioni proposte per l'integrale definito di una funzione limitata o no. Palermo, Virzì, 1914. 162 pp.

RICHARDSON (R. P.) and LANDIS (E. H.). Fundamental conceptions of modern mathematics. Chicago, Open Court, 1915. $1.25

SALMON (G.) und FIELDER (W.). Analytische Geometrie der Kegelschnitte. Neu herausgegeben von F. Dingeldey. In 2 Teilen. 1ter Teil. 8te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 30+452 pp. Geb. M. 12.00

SCHMIDT (G. C.). See LORENTZ (H. A.).

SKOLEM (T.). See BIRKELAND (K.).

SMITH (D. E.). See DEMORGAN (A.).

STUCKENBERG (K.). Elliptische Wurzelfunktionen. Strassburg, 1913. 8vo. 57 pp.

VOGT (H.). See ENCYCLOPÉDIE.

WEBER (H.). See ENCYCLOPÉDIE.

II. ELEMENTARY MATHEMATICS.

BIBLIOGRAPHY of education for 1911–12. (United States Bureau of Education Bulletin No. 657). Washington, Government Printing Office, 1915. 151 pp. $0.20

BRESLICH (E. R.). First-year mathematics for secondary schools. 3d edition. Chicago, University of Chicago Press, 1915. 8vo. 26+344 pp. Cloth. $1.00

CARCANO (C.). Matematica semplificata pratica, per scuole commerciali, industriali, licei, istituti tecnici, capomastri, ingegneri, professionisti; numerosi esercizi pratici completamente risolti. Vol. 3 (Progessioni geometriche). Milano, tip. Pivola e Cella, di P. Cella, 1915. 16mo. 63 pp. L. 1.00

CARSON (G. ST. L.) and SMITH (D. E.). Elements of algebra. Part 2. London, Ginn, 1915. Pp. 5+325–538. 2s. 6d.

——. Plane Geometry. 2 parts. London, Ginn, 1915. 6+266+6+224 pp. 2s. 6d.+2s. 6d.

GIEBEL (K.). Die Anfertigung mathematischer Modelle für Schüler mittlerer Klassen. (Mathematische Bibliothek No. 16). Leipzig, Teubner, 1915. 8vo. 4+52 pp. Boards. M. 0.80

GODFREY (C.) and PRICE (E. A.). Arithmetic. Parts 1–3 complete, with answers. Cambridge, University Press, 1915. 13+467 pp. 4s.

HADAMARD (J.). Leçons de géométrie élémentaire. I. Géométrie élémentaire. 5e édition. Paris, Colin, 1913.

KÖNNEMANN (W.). Rationale Lösungen von Aufgaben aus dem Gebiete der gesamten Elementarmathematik in funktioneller Abhängigkeit. Berlin, Winckelmann, 1915. 8vo. Geb. 8+112 pp. M. 3.00

LENNES (N. J.). See SLAUGHT (H. E.).

M'LAREN (G. C.). Improved four-place logarithm-tables. Multiplication and divison made easy. Cambridge, University Press, 1915. 27 pp. 1s. 6d.

MANCINELLI (F.). Alcune osservazioni di trigonometria piana. Forma probabile del triangolo e del quadrangolo. Problema fondamentale della planimetria. Pavia, tip. Popolare, di P. Mozzaglia, 1915. 8vo. 7+4+7 pp.

MERRIMAN (M.). Mathematical tables for class-room use. New York, Wiley, 1915. 8vo. 68 pp. $0.50

MONROE (P.). See SMITH (D. E.).

NORMAN (F. K.). See NORMAN (J. S.).

NORMAN (J. S. and F. K.). Norman's arithmetic for schools. London, Year Book Press, 1915. 8vo. 16+278 pp. 2s. With answers, 2s. 6d.

PRICE (E. A.). See GODFREY (C.).

REED (H. L.). Plane trigonometry. London, Bell, 1915. 13+290+16 pp. 3s. 6d.

SLAUGHT (H. E.) and LENNES (N. J.). Elementary algebra. Boston, Allyn and Bacon, 1915. 8vo. 10+357 pp. $1.00

SMITH (D. E.). Problems about war: for classes in arithmetic. With an introduction by Paul Monroe. New York, The Carnegie Endowment for International Peace, 1915. 23 pp.

——. See CARSON (G. ST. L.).

III. APPLIED MATHEMATICS.

ANDREWS (E. S.). An introduction to applied mechanics. Cambridge University Press, 1915. 8vo. 4s. 6d.

ANNUAIRE pour l'an 1915 publié par le Bureau des longitudes avec une notice scientifique par M. G. Bigourdain. Paris, Gauthier-Villars, 1915. 7+764+173+58 pp. Fr. 1.50

ANSEL (E. A.). Beiträge zur Dynamik und Thermodynamik der Atmosphäre. Göttingen, 1913. 8vo. 67 pp.+4 Tafeln. M. 2.00

BIGOURDAIN (G.). See ANNUAIRE.

BLOCK (E.). Ueber die Schmelzkurven einiger Stoffe. Göttingen, 1913. 8vo. 40 pp.

BOLZA (H.). Anwendung der Theorie der Integralgleichungen auf die Elektronentheorie und die Theorie der verdünnten Gase. Göttingen, 1913. 8vo. 36 pp.

BRUNNER (W.). Dreht sich die Erde? (Mathematische Bibliothek No. 17.) Leipzig, Teubner, 1915. Klein 8vo. 4+52 pp. Boards. M. 0.80

BURNHAM (R. W.). Mathematics for machinists. New York, Wiley, 1915. 8vo. 7+229 pp. $1.25

CASPAR (E.). Neue Methode zur Messung kleiner Temperaturdifferenzen, angewandt auf die Untersuchung der Temperaturverhältnisse in Kältemischungen mit Kohlensäureschnee. Marburg, 1913. 8vo. 68 pp.+1 Tafel.

CHAFFEE (E. L.). Physical laboratory manual. Cambridge, Mass., 1914. 8vo. 128 pp. $1.50

CLIFFORD (W. G.). See REECE (T.).

CRAPPER (E. H.). Arithmetic of alternating currents. London, Whittaker, 1915. 7+208 pp. 2s. 6d.

DALE (R. B.). Arithmetic for carpenters and builders. New York, Wiley, 1915. 8vo. 6+228 pp. $1.25

DARWIN (E.). Emma Darwin: a century of family letters, 1792–1896. Edited by her daughter Henrietta Litchfield. 2 volumes. London, Murray, 1915. 31+289+25+326 pp. 21s.

DONNER (G.). Ueber die Stabilität der quasi-beständigen Bewegung. Strassburg, 1913. 8vo. 44 pp.

DUCHÊNE (—.). The mechanics of the aeroplane, by Captain Duchêne. Translated from the French by J. H. Ledeboer and T. O'B. Hubbard. New edition. New York, Longmans, 1915. 10+230 pp. $2.25

DUHEM (P.). La science allemande. Paris, Hermann, 1915. 146 pp.

EGGERT (O.). See JORDAN (W.).

FAWDRY (R. C.). Statics. Part 2. London, Bell, 1915. Pp. 5+159–305+8. 2 s.

HAAG (H.). Die Geschichte des Nullmeridians. Giessen, 1913. 8vo. 111 pp.+1 Karte.

HENDERSON (R.). Mortality laws and statistics. (Mathematical monograph series, No. 15.) New York, Wiley, 1915. 8vo. 6+111 pp. Cloth. $1.25

HUBBARD (T. O'B.). See DUCHÊNE (—.).

JAHNKE (E.). See RUNGE (C.).

JOHANSEN (N. P.). Astronomisk Bestemmelse af Laengdedifferenzen mellem Kjöbenhavns Observatorium og Buddinge samt af Azimuthes i Buddinge af Retningen mod Nikolaj Taarn. Kjöbenhavn (Danske Gradmaal), 1914. 4to. 120 pp. M. 5.00

JOHNSON (J. F.). Practical shop mechanics and mathematics. New York, Wiley, 1915. 8vo. 7+130 pp. $1.00

JONES (H. S.). Numerical examples in physics. London, Bell, 1915. 8vo. 12+332 pp. 3s. 6d.

JORDAN (W.). Handbuch der Vermessungskunde von W. Jordan fortgesetzt von C. Rheinhertz. Band 2: Feld- und Landmessung. 8te erweiterte Auflage bearbeitet von O. Eggert. Stuttgart, J. B. Metzler, 1914. 8vo. 10+938+55 pp. M. 22.50

JOURDAIN (P. E. B.). See MACH (E.).

KRÖNCKE (H.). Ueber die Messung der Intensität und Härte der Röntgenstrahlen. Göttingen, 1913. 8vo. 55 pp.

LECORNU (L.). Cours de mécanique professé à l'Ecole polytechnique. Tome 2. Paris, Gauthier-Villars, 1915. 6+538 pp. Fr. 18.00

LEDEBOER (J. H.). See DUCHÊNE (—.).

LILLY (S. B.). See MILLER (J. A.).

LITCHFIELD (H.). See DARWIN (E.).

LUCY (A. W.). Exercises in laboratory mathematics. Oxford, Clarendon Press, 1915. 245 pp. 3s. 6d.

LYNDE (C. J.). Physics of the household. New York, Macmillan, 1915. 8vo. 315 pp. 5s. 6d.

MACH (E.). The science of mechanics: a critical and historical account of its development. Supplement to the third English edition translated and annotated by P. E. B. Jourdain. London and Chicago, Open Court, 1915. 14+106 pp. 2s. 6d.

MANN (H. L.). A text-book on practical mathematics for advanced technical students. London, Longmans, 1915. 12+488 pp. 7s. 6d.

MILLER (H. W.). Descriptive geometry. 3d edition. New York, Wiley, 1915. 8vo. 149 pp. $1.50

MILLER (J. A.) and LILLY (S. B.). Analytic mechanics. Boston and New York, Heath, 1915. 15+297 pp. $2.00

PARKER (G. W.). Elements of optics for the use of schools and colleges. London and New York, Longmans, 1915. 8vo. 122 pp. $0.75

REECE (T.) and CLIFFORD (W. G.). Billiards. London, A. and C. Black, 1915. 8vo. 7s. 6d.

RHEINHERTZ (C.). See JORDAN (W.).

RUNGE (C.). Graphische Methoden. Uebersetzung der Vorlesungen, die im Winter 1909–10 an der Columbia-Universität gehalten waren. (Sammlung mathematisch-physikalischer Lehrbücher herausgegeben von E. Jahnke, Band 18.) Leipzig, Teubner, 1915. 8vo. 4+142 pp. Geb. M. 5.00

SCHUR (F.). Vorlesungen über graphische Statik. Unter Mitwirkung von W. Vogt. Leipzig, 1915. Gr. 8vo. 8+219 pp. M. 7.00

STARK (J.). Prinzipien der Atomdynamik. Teil 3: Die Elektrizität im chemischen Atom. Leipzig, 1915. 8vo. 16+280 pp. M. 8.00

TRAUB (G.). Ueber die Vertical-Geschwindigkeitskurve. Untersuchungen über die Verteilung der Wassergeschwindigkeiten im offenen Wasserlauf. Dresden, 1913. 8vo. 160 pp.+6 Tafeln. M. 4.00

VALIER (M.). Das astronomische Zeichnen. Leichtfassliche und gemeinverständliche Anleitung zur Beobachtung und zeichnerischen Darstellung cölestischen Objekte nach dem Anblick im Fernrohr für Laien und Amaturastronomen. Mit Anhang: Mondaufnahmen. München, 1915. 8vo. M. 1.50

VOGT (W.). See SCHUR (F.).